Quantum memory: influence of the noise on the storage and retrieval process

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Résumé

Mémoire quantique : influence du bruit sur les processus de stockage et de récupération

Le stockage de photons dans un ensemble atomic est un sujet particulièrement important pour le domaine des communications quantiques et de la cryptography quantique. Les photons sont en effet de bons vecteurs de l'informations quantique. Cependant le traitement de l'information nécessite leur stockage dans une mémoire. Des recherches sur l'optimisation des processus de stockage et de récupérations ont déjà été effectuées. En partant de ces travaux, nous avons étudié l'influence du bruit sur les processus mis en jeu. Nous avons modélisé la mémoire par un système Λ à trois états, et avons considéré son interaction avec un laser duquel sont dérivés le champ de contrôle et l'impulsion entrante. Nous avons ensuite établis les équations donnant l'expression de l'efficacité en fonction de l'intensité pour deux types de bruit différents : du bruit en amplitude et en fréquence.

Les résultats analytiques sont développés au premier ordre en intensité du bruit et comparés à des simulations. De plus nous montrons qu'en présence de bruit en amplitude peut être optimisé en modifiant l'enveloppe de l'impulsion à stocker connaissant l'intensité du bruit. D'après les résultats obtenus, pour le modèle de bruit considéré, l'efficacité est robuste.

Abstract

Quantum memory: influence of the noise on the storage and retrieval process

The storage of photon in an atomic media is a topic of great interest for quantum communications including quantum cryptography. In fact, photons are good quantum information carriers. However processing the information requires to store them locally in a memory. Research concerning the optimization of storage and retrieval processes for quantum memories have already been achieved. Starting with this work, the question was to study what happened in case there was some noise that disturbed the storage and retrieval processes. So using a three level Λ system as the memory model, and considering a laser as a single source which produce the control field and the probe pulse, we derived the relations that give the efficiency as function of the noise intensity for two types of noise in the laser : amplitude noise and frequency noise.

The analytical results are given to first order in noise intensity and are compared to simulated results. Furthermore, we show that the efficiency in presence of amplitude noise can be optimized by shaping the probe pulse as function of the noise intensity. According to the results we obtained, the efficiency is very robuste to the kind of noise we have considered.

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Glossary

1. AOM (Acousto Optic Modulator): The AOM is a device that uses sound waves to modulate the shape and the frequency of an electromagnetic signal by modifying the optical index of a medium which is crossed by the laser pulse to modulate.

2. Coherent state: a coherent state $|\alpha\rangle$ is the eigenstate of the annihilation operator $\hat{a}$ of the quantum harmonic oscillator system $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$. This is the quantum light state which is the closest to the classical light sinusoidal wave representation.

3. EOM: Electro Optic Modulator uses electric current to modulate the shape and the frequency of an electromagnetic signal.

4. Homodyne measurement: an homodyne measurement is a method for detecting frequency modulated signal by making it interfer with a reference signal that comes from the same source.

5. NBI: The Niels Bohr Institute

6. Probe pulse: We use probe pulse as a weak incoming signal we wish store in a quantum memory.

7. QUANTOP: Danish National Research Foundation Center for Quantum Optics

1 Introduction

Quantum information and quantum optics are fields that aim to use the most fundamental properties of quantum mechanics to design new computation and communication technologies. Quantum communications work experimentally and some firms also commercialize quantum cryptography devices [17]. However, the ability to perform long distance quantum communication (more than one hundred kilometers) is still the subject of research. The main promising approach is the use of quantum repeaters [12, 11]. Such device requires quantum memory to work. That is why quantum memory is a topic on which several laboratories around the world are working on [8, 5, 6, 7, 8].

This report is organized as follow. First, we give some informations about the Niels Bohr Institute where two groups that belong to the Danish National Research Foundation Center for Quantum Optics (QUANTOP) are working. Then we give a short introduction to quantum cryptography and explain why quantum memories are required to improve the communication distance. After this, we give the model that we use for quantum memory and its principle. Then we focus on the effect of the storage and retrieval efficiency of the memory assuming there is noise.

Figure 1: The Niels Bohr Institute

2 The Niels Bohr Institute

2.1 History

The Institute of Theoretical Physics of the University of Copenhagen has been founded in 1921 thanks to the famous Danish physicist Niels Bohr. During the 1920s, and 1930s, the Institute was the concordant crossroad of the developing disciplines of atomic physics and quantum physics. Physicists from across Europe (and sometimes further abroad) often visited the Institute to confer with Bohr on new theories and discoveries. The Copenhagen interpretation of quantum mechanics is named after work done at the Institute during this time. On Niels Bohr's 80th birthday - October 7, 1965 - the Institute for Theoretical Physics of the University of Copenhagen officially became The Niels Bohr Institute.

2.2 Organization

The Niels Bohr Institute belongs to the Faculty of Science of the Copenhagen University. The Niels Bohr Institute is involved in research and education within astronomy, geophysics, nanophysics, general physics and biophysics.

The institute is situated around 'Fælledparken' at three different locations: Rockefeller Komplekset at Juliane Maries Vej 26-32, H.C. Ørsted Institutet at Universitetsparken 5 and Blegdamsvej 17-21.

There are 76 scientific staff members, 74 technical and administrative staff members, 69 PhD students, and a large number of international researchers and students.

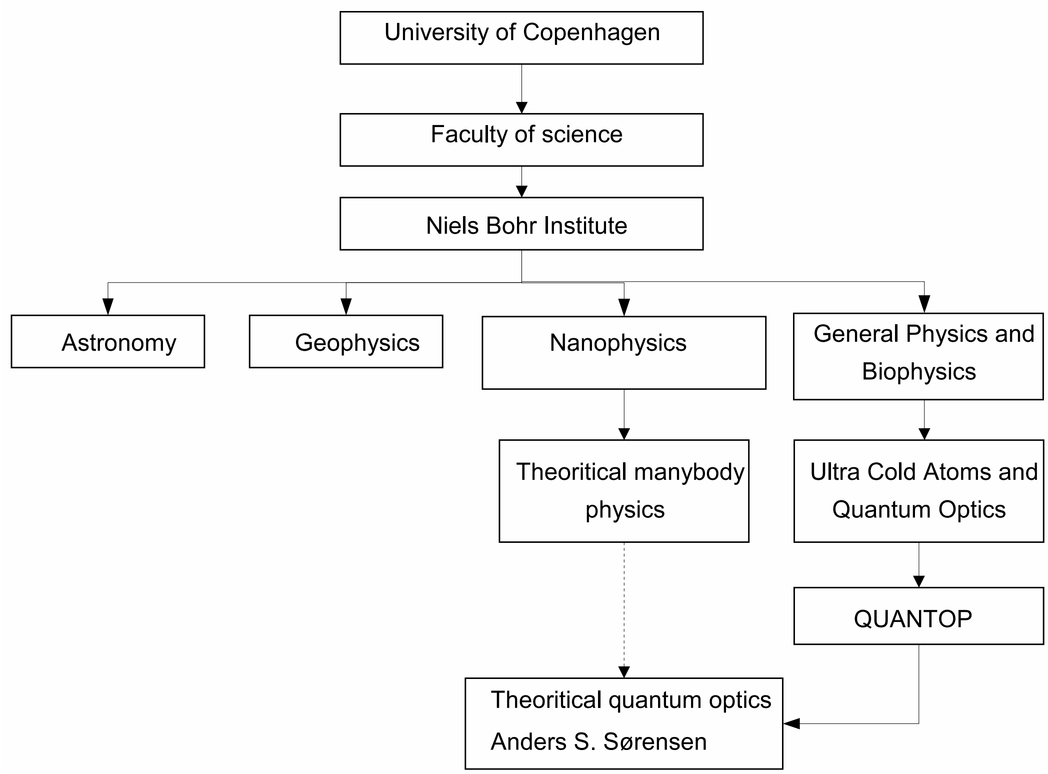

Figure 2: Organizational chart

2.3 QUANTOP (Danish National Research Foundation Center for Quantum Optics)

Danish National Research Foundation Center for Quantum Optics - QUANTOP was founded in 2001 and is funded until 2012. The Center involves research groups at the Niels Bohr Institute of the University of Copenhagen and at the Department of Physics and Astronomy at the University of Aarhus. Eugene Polzik is the director of the Center.

The research activities of the Center are carried out by four groups:

- Quantum Optics Lab (Copenhagen)
- Quantum Theory Group (Copenhagen)
- Ion Trap Group (Aarhus)
- Quantum Gas Lab (Aarhus)

The research concentrates on quantum state engineering for light, atoms and ions, including entangled, squeezed and other interesting states. Quantum information processing, including quantum computing, quantum teleportation, quantum cryptography and quantum memory, is one of the major directions of the research. Studies of ultra-cold atoms, both fundamental research on its properties and dynamics, and applications for quantum information processing and precision measurements are carried out theoretically and experimentally.

3 Quantum cryptography

Quantum cryptography is a practical application of the field of quantum communication and the technology is already available and commercialized.

3.1 Principle

The principle of quantum cryptography is to use a one-time-pad combined with quantum mechanics to share the key. A one-time-pad is a cryptography method that allows to send only one time a message with a key which has at least the lenght of the message. This key is randomly generated and then is shared between Alice and Bob. If Alice want to send one binary message to Bob, she just need to make the addition between the message and the key. Then she sends the encrypted message to Bob. And Bob just make the addition between the encrypted message and the key. Then the key has to be discarded and not used again.

$$Encryptedmessage = message \oplus key$$

$$Decryptedmessage = Encryptedmessage \oplus key$$

where $\oplus$ stands for the bit to bit addition in the sense $0 \oplus 1 = 1 \oplus 0 = 1$ and $1 \oplus 1 = 0 \oplus 0 = 0$.

It is the most provable secure way to send a message to someone. However the main problem is the ability to share the key. This is where quantum mechanics has a role to play. There is a protocole that requires one classical public channel and one quantum channel that allows to share the key.

Before explaining the protocole in itself, let us tell some words about the quantum channel. Here we assume that the quantum channel is perfect. Alice has a perfect single photon source with an Electro Optical Modulator (EOM) that can be seen as a switch that allows to turn on and off a half wavelength plate. Bob has a polarizer. Since the quantum channel is ideal, there are no absorption of photon or loss of polarization in the channel. The EOM makes possible to rotate the polarization of the photon by 45°. The photon can thus be sent in two differents basis (figure 3). The polarizer can be switched into two different positions. These positions allow to make the measurement of the photon in the two different bases. In each bases, there are two orthogonal states. $\langle H|V\rangle = 0$, $\langle -45|45\rangle = 0$. The overlap of any two states of differents bases is such that they give equal probabilities : $|\langle -45|H\rangle|^2 = |\langle -45|V\rangle|^2 = |\langle 45|V\rangle|^2 = |\langle 45|H\rangle|^2 = \frac{1}{2}$. In order to make the correspondance with binary information, we choose the convention that the states $|H\rangle$ and $|45^{\circ}\rangle$ stand for $|1\rangle$ and the states $|V\rangle$ and $|-45^{\circ}\rangle$ stand for $|0\rangle$.

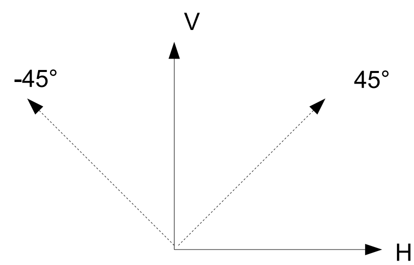

Figure 3: The two polarisation bases

Let us now explain the protocol.

- Alice send photons to Bob. Each photon is sent in one of the two bases in one state of this bases $|0\rangle$ or $|1\rangle$. The choice of the bases is done by a random number generator. The state of the photon $|0\rangle$ or $|1\rangle$ is also chosen with a random generator. For each photon that Bob receives, he makes a measurement in one of the two bases. The choice of the bases is done by a random number generator that is not correlated to Alice's one. If no ones try to make a measurement between Bob and Alice, the error rate in the raw key is 25%.

- Bob and Alice then announce the bases they have chosen with the public channel. They then only keep the bits that have been measured in the same bases as they have been sent. After having discarded the bits which were measured in the wrong bases, a shorter key is obtained.

- By announcing some of the bits they have kept, they can then compare if they are the same. If this is the case, this means that no one has tried to intercept the key. They can for instance choose randomly 50% of the bits of the shorter key. If they all matches, they keep the half of the key that remains to encrypt the message. In fact the remaining key must be at least of the length of the message.

- The message is then encrypted using the remaining key. This is a classical one-time-pad.

We now just say a few words about the eavedropping. If an eavedropper, Eve, attempts to get the key , Eve will need to measure some photons from Alice and then send the same number of photon to Bob. To make the measurement, Eve need to choose a bases. Since she does not know Alice's bases, she will choose it randomly. But Eve does not know wether she choses the good bases or not and so she sometimes send photons to Bob in the wrong bases. If in this case Alice and Bob have the same bases. When they compare the shorter key, they find some differences.

If someone attempts to intercept the communication, then there will be difference in some bits which belong to the shorter key. The only thing to do is in this case to discard the whole key. In practice however, there are many other considerations to take into account : all kind of error that occurs, statistical analysis, other kind of protocols . But the main principle remains the same : using quantum mechanics to share the key. Much more informations about this subject can be found in the paper [15].

3.2 Quantum repeaters

Today, the main obstacle to the commercialization at big scale of quantum cryptography devices is its implementation on large distances. In practice, loss of photons increase with the distance. Quantum repeaters are devices that use property of entanglement swapping to teleport the state of a photon to another distant photon. And such device could in principle enhance dramatically the distance on which quantum key distribution can be done.

Let us now tell a few words about entanglement swapping. A system of two photons is called maximally entangled if measuring one of these photons make the other one in a determinate state, and if the single particle density is completely mixed (which means the probability to measure one state or the other are the same). For instance $\frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle+|1\rangle|0\rangle\right)$ is a maximally entangled system. The entanglement swapping is the ability to make two separated system entangled without making them interact. For instance let us consider three persons Alice, Bob and Claire. Alice and Bob share an entangle state $\frac{1}{\sqrt{2}}|0_A\rangle|0_{B1}\rangle+\frac{1}{\sqrt{2}}|1_A\rangle|1_{B1}\rangle$. Bob and Claire share $\frac{1}{\sqrt{2}}|0_{B2}\rangle|0_C\rangle+\frac{1}{\sqrt{2}}|1_{B2}\rangle|1_C\rangle$. Where the index A stands for Alice's photon state, $B1$ and $B2$ stands for Bob's photons states and C stands for Claire's photon state. Then if Bob perform a special kind of measurement (called Bell's measurement) that involves the photons $B1$ and $B2$, the photons of Alice and Claire are then maximally entangled $\frac{1}{\sqrt{2}}|0_A\rangle|0_C\rangle+\frac{1}{\sqrt{2}}|1_A\rangle|1_C\rangle$ even if they have not interacted. This is what is called entanglement swapping.

One important problem is how to perform a measurement that involves two photons? The main approach is to store locally in a memory the states of the two photons, and then to perform the measurement by processing the information in the quantum memory.

4 Quantum memory

This section presents the model that is used for the quantum memory and what are the efficiency of storage and retrieval of photons that can be achieved.

4.1 Model

4.1.1 Main idea

The quantum memory is basically a three states quantum system. One state is an excited state $|e>$ with a short life time. The two others states are a ground state $|g>$ and a metastable state $|s>$. The long life time of the state $|s>$ is due to the fact that there is no electric dipole allowed transition between $|s>$and $|g>$. The main idea is to couple this system with two electromagnetic fields: the probe pulse which contains the information that we want to store, and the control field which will help storing the probe pulse. The probe pulse is a weak pulse, with few photons. Since we are interested in the quantum information of the pulse, it will be described by an operator. On the contrary, the control field is a strong pulse described in a classical manner.

At the end of the storage process, all the information is contained in a superposition of $|g>$ and $|s>$. It follows that the relevant operator to describe the information that has been stored is in fact $|s><g|$. Of course, it is impossible to store anything on a transition involving the excited state because of the fast decay. One can argue that in practice there is also a decay due to higher order moment transitions between $|g>$ and $|s>$. However what we need is only to store the information during the time we need to process it. And this can be achieved in this way. The retrieval of the photon is done by applying a control field after the storage process.

In theory, the memory can work with only one atom. The three states are those which can be reached by a single electron. The storage process for a coherent probe pulse in one is shawn on figure $\{4\}$. However, the coupling constant g between one atom and one photon is weak. In such case, the storage can only be done by using extremely good cavities. Such cavities are however very hard to build.

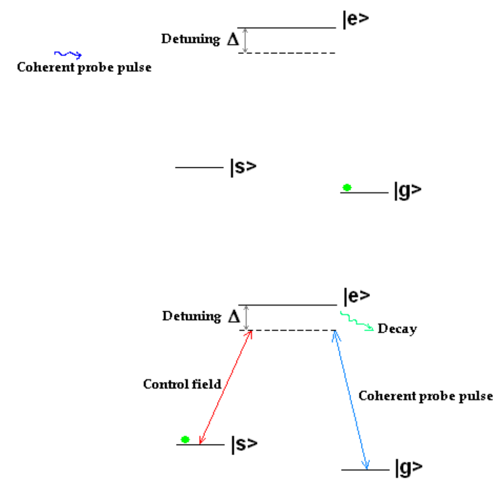

Figure 4: Storage of a coherent probe pulse in one atom.

The figure {4} only shows what happens for a coherent state. But the principle remains the same for a superposition of coherent states. For instance, sending one photon on a beamsplitter gives a superposition of zero and one photon on each path of the beam splitter. One can in principle store the superposition of zero and one photon in the quantum memory as a superposition of $|g>$ and $|s>$. At the moment, experiments for storage have not only been achieved for coherent pulses [13], but also for superpositions of states [14].

Although a quantum memory works in principle with a single atom, in practice, what is used is an atomic ensemble of N atoms. This has only advantages compared to the case of a single atom.

- The memory is experimentally simpler to build since it is harder to select only one atom.
- The effective coupling constant between the atomic ensemble and the quantum electromagnetic field is enhanced by a factor $\sqrt{N}$ so that the effective coupling constant is given by $g\sqrt{N}$.

We furthemore assume that the atomic ensemble is surrounded by a cavity. In this case, there is only one frequency mode for the probe pulse and the equations are simpler compared to the case of the free space.

4.1.2 Operators and notations

In order to describe the storage and retrieval processes, some collective operators are needed. Here we will only give and explain the role of these operators and give the master equations that they follow. We need to describe the $|s> - |g>$ transitions, the $|g> - |e>$ transitions, and the probe pulse.

- The S operator describes the collective $|s><g|$ transitions for all atoms. We call it spinwave annihilation operator. This is the one that describes the stored information.
- The P operator describes the $|e><g|$ transitions for all atoms. We call it polarisation operator.
- The ξ_{in} operator describes the shape of the probe pulse during the storage.
- The ξ_{out} operator describes the shape of the output pulse during the retrieval.
- The classical control field is described by its Raby frequency Ω (Annexe A).

The control field and the probe pulse are monomode.

- The probe pulse frequency is ω_1.
- The control field frequency is ω_2.

However, we prefer to describe the frequency of these fields respect to the relevant atomic transitions.

$\omega_1 = \omega_{eg} - \Delta_1$ where Δ_1 is the detuning between the probe pulse frequency and the $|e> - |g>$ resonant frequency.

$\omega_2 = \omega_{es} - \Delta_2$ where Δ_2 is the detuning between the probe pulse frequency and the $|e> - |s>$ resonant frequency.

In all that follow, we will consider that these detuning are the sames. $\Delta_1 = \Delta_2 = \Delta$.

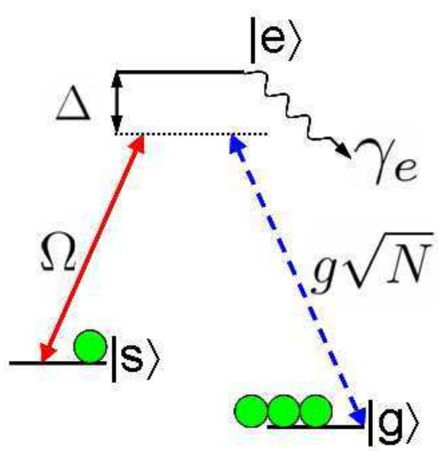

Figure 5: Λ atomic media scheme. The blue dash line is the quantum field. The red line is the classical field. [2]

We now need a few more physical quantities to write down the equations of motion for the operators.

- γ is the total decay of the optical transition $\sigma_{eg} = |e><g|$ including the decay due to spontaneous emission γ_e .
- κ is the constant that caracterize the leakage of the pulse due to the cavity.
- We can now define $C = \frac{g^2 N}{\kappa\gamma}$, the cooperativity parameter of the cavity.

The master equations are derived from the Hamiltonian H of the system using the equations of motion $\frac{dA}{dt} = i[A, H]$ where A stands for any operator. In the interaction picture, $H = \hbar\Delta\sigma_{ee} - (\hbar\Omega(t)\sigma_{es} + \hbar g\xi\sigma_{eg} + h.c)$ where $\sigma_{ij} = |i><j|$ and $h.c$ stands for hermitian conjugate. However, this hamiltonian does not take into account the decay between the states $|e\rangle$ and $|g\rangle$, $|e\rangle$ and $|s\rangle$, and $|g\rangle$ and $|s\rangle$. Since the transition between $|g\rangle$ and $|s\rangle$ is not dipole allowed, we consider that this decay is long enough compared to the time we want to store the photon. We thus only consider the decay of the excited state γ. Using the assumption that all the atoms are in the ground state at the beginning, and that there are almost always most of the atoms in the ground state, taking into account the decay γ ,the master equations can be written as:

$$\begin{aligned}
\xi_{out} &= \xi_{in} + i\sqrt{2\gamma C}P \\
\dot{P} &= -(\gamma(1+C) + i\Delta)P + i\Omega S + i\sqrt{2\gamma C}\xi_{in} \\
\dot{S} &= i\Omega^{\star} P
\end{aligned} \tag{1}$$

All the details are given in the paper [2].

4.2 Experimental setup

We are going now to describe a typical experimental setup. This will be required when we will add noise in the control field. In fact, the result that we will obtain will be closely linked to the setup.

4.2.1 Storage

Our model is to derive the control field and the probe pulse from the same laser. This may not be the case in industrial implementations, however this is the case in most experiments which are performed on quantum memory. The required shape for the probe pulse is realized by an Acousto Optic Modulator. The Rabi frequency is chosen by a second AOM which shapes $E(t)$. The storage is then achieved in an atomic ensemble surrounded by a cavity.

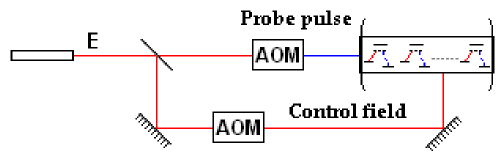

Figure 6: Experimental setup for the storage of a probe pulse

The storage starts at time $t = 0$ and stops at $t = T$.

4.2.2 Retrieval

In practice, if there is a spinwave mode stored in the memory, applying a control field will result in the retrieval of an output pulse. However, we need to have some device to check the shape of the output mode. This is done by homodyning the output pulse with an other signal that is generated from the same laser that is used as control field for retrieval. The frequency shift of the homodyning pulse is done by an AOM. The shaping of the control field is done by an AOM too.

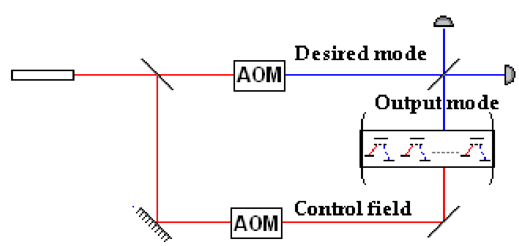

Figure 7: Experimental setup for the retrieval of a probe pulse

The retrieval starts at $T_R > T$. However for all the calculations, we will use $T_R = 0$ when we will consider only retrieval since it does not change anything.

4.3 Storage and retrieval efficiency

The total efficiency of the memory is given by the probability to store a photon and retrieve it in the same mode. The following results will be given assuming we are allowed to eliminate adiabatically the polarization. This means that we put $\dot{P} \approx 0$ in the relations (1). It has been proved that it is allowed when $TC\gamma \gg 1$ [2].The whole details concerning the two following sub-sections are explained in [2].

4.3.1 Storage efficiency

The storage efficiency is defined by the number of stored excitations divided by the number of incoming photons. Without loss of generality, we renormalize the probe pulse to have one photon. The efficiency is then given by the number of excitations. Since S has all the properties of an annihilation operator [2], the number of stored spinwave is given by the usual number operator $SS^{\dagger}$. We are not interested in each eigenvalue of the operator but only in the mean value. In this case, the S operator can be considered as a complex number. The storage efficiency is written as

$$\eta_s = SS^{\star} = |S(T)|^2$$

Adding the assumption that $\xi_{out} = 0$ since we are interesting in storage, one can solve the two last equations of the system (1) and find that S operator is then given by (2).

$$S(T) = \sqrt{\frac{C}{1+C}} \int_0^T f(t)\xi_{in}(t)dt \tag{2}$$

where $f(t)$ is the function defined by

$$f(t) = -\Omega^{\star}(t)\frac{\sqrt{2\gamma(1+C)}}{\gamma(1+C)+i\Delta} e^{-\frac{h(t,T)}{\gamma(1+C)+i\Delta}} \tag{3}$$

and $h(t_1, t_2)$ is given by

$$h(t_1, t_2) = \int_{t1}^{t2} |\Omega(t)|^2 dt \tag{4}$$

Let us just give some physical meaning about the phase factor in the exponential. This phase factor $\frac{i\Delta h(t,T)}{\Delta^2+\gamma^2(1+C)^2}$ is called the AC-stark shift and it correponds to the phase that is added to the probe pulse due to its interaction with the atomic ensemble.

Now the whole problem is to maximise the storage efficiency. It has been proved that with the assumptions we have used, it is always possible to reach a maximum efficiency equal to $\frac{C}{1+C}$. The maximal efficiency is obtained if the following condition is fullfilled

$$\xi_{in}(t) = f(t)^{\star}. \tag{5}$$

We first see that this condition compensates the AC-stark shift since it is the complex conjugate of f. One way to show that this condition gives the maximal efficiency is to notice that $f(t)$ is normalized. This can be written $\int_0^T f(t)f(t)^{\star}dt = 1$. Since we want ξ_{in} to be normalized too and reminding that the integral over the product of two normalized functions $a(t)$ and $b(t)$ is a scalar product, the Cauchy-Schwarz inequality gives $< a|b > \leq 1$ so that $\int_0^T f(t)\xi_{in}(t)dt \leq 1$. This integral is equal to one if and only if the condition (5) is fullfilled. Since we know the optimized shape for the probe pulse, we can then compute the storage efficiency and obtain

$$\eta_s = \frac{C}{1+C}\left[1 - e^{-\frac{2\gamma(1+C)}{\gamma^2(1+C)^2+\Delta^2}h(0,T)}\right]^2 \tag{6}$$

which gives $\eta_s \approx \frac{C}{1+C}$ provided $h(0,T)$ is big enough. Getting $h(0,T)$ big requires to apply a control field during a sufficiently long period, or a control field with a strong enough amplitude.

4.3.2 Retrieval efficiency

The retrieval efficiency is defined by the number of retrieved photons divided by the number of stored excitations. Without loss of generality, we renormalize the spinwave to have one stored excitation ($S(0) = 1$). The efficiency is then given by the number of retrieved photons into the desired output mode. This is given by $\eta_r = AA^\star = |A|^2$where

$$A = \int_{T_R}^{\infty} \xi_{out}(t)\xi^\star_{desiredoutput}(t)dt \tag{7}$$

Here we just choose the desired mode equal to the output mode to get the maximal efficiency $\xi_{desiredmode} = \xi_{out}$.

With the adiabatic elimination of P and assuming that $\xi_{in} = 0$ since we are interested in retrieval process, solving the master equations (1) gives the following result

$$\xi_{out}(t) = -\sqrt{2\gamma C}\frac{\Omega(t)}{\gamma(1+C)+i\Delta}e^{-\frac{1}{\gamma(1+C)+i\Delta}h(0,t)}. \tag{8}$$

The retrieval efficiency is then given by

$$\eta_r = \frac{C}{1+C}\left[1 - e^{-\frac{2\gamma(1+C)}{\gamma^2(1+C)^2+\Delta^2}h(0,\infty)}\right]^2. \tag{9}$$

The maximal retrieval efficiency can be achieved provided the control field is applied for a sufficiently long period or provided the control field is strong enough (that means to have $h(0,\infty)$ big enough do drop the exponential).

4.4 Time reversal

Despite the system contains an irreversible decay γ, time reversal is still an important concept. It has been proved in the paper [3] that knowing the optimal retrieval strategy gives the optimal storage strategy using the time reversal concept. It works both for cavity and free space. And it shows that the optimal incoming mode is the time-reverse of the optimized retrieved mode. Moreother, the optimal storage control Ω_r is the time-reverse of the retrieval control Ω_s,

$$\Omega_r(t) = \Omega^\star_s(T-t). \tag{10}$$

In that case, the output pulse is simply linked to the input pulse by the relation:

$$e(t) = \xi^{\star}_{int}(T - t) \tag{11}$$

where $e(t)$ stands for the output pulse which is renormalized to contain exactly one photon. The results that we have just given previously fullfill this relation. It is also time reversal that explain why the storage efficiency is equal to the retrieval efficiency.

5 Noise in quantum memory

We have seen that in certain limit the maximal storage and retrieval efficiency. This has been done assuming a perfect probe pulse and a perfect control field. In practice there is always some noise. The whole problem that we are interested in is the behavior of the efficiency in such conditions. We will consider two different types of noise : amplitude noise and phase noise.

5.1 Amplitude noise

5.1.1 Noise model

The amplitude noise is a random fluctuation of the amplitude of the electromagnetic field of the laser which is used for both the probe pulse and the control field. We will assume that the noise is white gaussian noise and that there is a feedback on the laser. We write the electric field as

$$E(t) = (E_0(t) + \delta E(t))e^{i\omega t} \tag{12}$$

where δE represents the noise. The given relation for δE taking the feedback into account is :

$$\dot{\delta E}(t) = A_N E_0 \eta(t) - \gamma_N \delta E(t) \tag{13}$$

- The strengh of the feedback is γ_N. The bigger is γ_N, the smaller is the noise amplitude.
- $\eta(t)$ is gaussian white noise normalized to have

  $$Autocorr(\eta(t)\eta(t')) = T\delta(t - t') \tag{14}$$

  where T is the duration of the period on which we have computed the autocorrelation.
- A_N is a normalization factor that gives the proportion of noise in the signal. For $A_N = 0$, there is no noise. For $A_N = 1$, there is as much noise as signal.

Solving the differential equation (13 gives the following result :

$$\delta E(t) = A_N E_0 \int_{-\infty}^{t} \eta(t') e^{-\gamma_N (t-t')} dt' \tag{15}$$

In order to compute the efficiency in the following parts, we will need the autocorrelation function for δE. Using the relations (15) and (14), we obtain the autocorrelation function

$$Autocorr(\delta E(t)\delta E(t')) = I_N T E_0^2 . exp(-\gamma_N |t - t'|), \tag{16}$$

where $I_N = \frac{A_N^2}{\gamma_N}$. What is interesting for us in the following is the instantaneous value of the noise intensity. This is given by

$$< \delta E(t)\delta E(t') >= \frac{Autocorr(\delta E(t)\delta E(t'))}{T}$$

$$< \delta E(t)\delta E(t') >= I_N E_0^2.exp(-\gamma_N\,|t-t'|).$$

5.1.2 theoretical results

The theoretical results have been obtained with the help of MAPLE (Annexe D.2).

Storage The point is now to compute the storage efficiency for the considered experimental setup. One can express the Rabi frequency and the shape of the probe pulse in function of $E(t)$

$$\Omega(t) = \alpha(t).E(t) \tag{17}$$

$$\xi_{in}(t) = \beta(t)E(t). \tag{18}$$

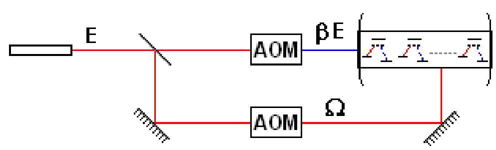

Figure 8: Storage process including noise

$\alpha(t)$ is the shape of the Rabi frequency which is given by the AOM and $\beta(t)$ is the shape of the probe pulse. Starting with there are two main ways to optimize the storage process :

- Optimizing the shape of the control field for a given probe pulse.
- Optimizing the shape of the probe pulse for a given control field.

This is the last approach that we will consider. We will now compute the efficiency using two differents shape for the probe pulse:

- The shape that maximizes the efficiency when we do not take into account the noise. We will refer it as the standard mode.

- The shape that maximizes the efficiency assuming we know something about the average of the noise. We will refer it as the optimized mode.

Since we consider noise which is a random process, the relevant quantities will be the average amplitude of the spinwave squared $< S >< S^\star >$ and the average number of excitations stored $< SS^\star >$ where $\langle\rangle$ stands for averaging over the noise.

Standard mode The shape $\beta(t)$ of this mode is already known since it is the one we have computed when there is no noise which is given by the relation (5)

$$\beta(t) = -\alpha(t)\frac{\sqrt{2\gamma(1+C)}}{\gamma(1+C) - i\Delta}e^{-\frac{h_0(t,T)}{\gamma(1+C)-i\Delta}} \tag{19}$$

with

$$h_0(t_1, t_2) = \int_{t1}^{t2} |\Omega_0(t)|^2 dt \tag{20}$$

where the index 0 means that there is no noise in the electromagnetic field. In practice, it means that when we compute the shape of the probe pulse, we will take the value of the Rabi frequency of the control field assuming there is no noise. Then, assuming there is one photon on average in the probe pulse,

$$\xi_{in} = -\Omega(t)\frac{\sqrt{2\gamma(1+C)}}{\gamma(1+C) - i\Delta}e^{-\frac{h_0(t,T)}{\gamma(1+C)-i\Delta}}\sqrt{\frac{1}{1+I_N}} \tag{21}$$

The $\sqrt{\frac{1}{1+I_N}}$ factor is the renormalization factor. It makes sure that we have on average one photon in the probe pulse $\int_0^T \langle\xi_{in}(t)\xi^\star_{in}(t)\rangle\, dt = 1$. If there is no noise the renormalization factor is equal to 1 which gives the same expression as in relation (5).

Using the definition of S, we will compute the average efficiency

$$< S >= \sqrt{\frac{C}{1+C}}\int_0^T < f(t)\xi_{in}(t) > dt.$$

However, computing the exact $< S >$ requires that we are able to compute $< \Omega(t)\Omega^\star(t)e^{-\frac{h(t,T)}{\gamma(1+C)+i\Delta}} >$ which is very hard. Considering this, the solution that we will give is expanded to the second order in the noise δE (and so to the first order in I_N). This solution requires that the noise must be low compared to the amplitude of the field $I_N \ll 1$.

In that case, we obtain the following expression for the amplitude of the spin wave squared

$$
\begin{aligned}
< S >< S^\star > \quad &= \tfrac{C}{1+C} \tfrac{1}{1+I_N} \int_0^T \quad |\Omega_0(t)|^2 |K_1|^2 e^{\left(-|K_1|^2 \int_t^T |\Omega_0(t')|^2 dt'\right)} \\
&\times \quad \left\{ 1 + I_n \left[2 - 4|K_1|^2 \int_t^T |\Omega_0(t')|^2 e^{(-\gamma_N(t'-t)} dt' - |K_1|^2 \int_t^T |\Omega_0(t')|^2 dt' \right.\right. \\
&+ \quad \left.\left. 4\left(|K_1|^4 - 2|K_0|^{-2}\right) \int_t^T \int_{t_1}^T |\Omega_0(t_1)|^2 |\Omega_0(t_2)|^2 e^{(-\gamma_N(t_2-t_1)} dt_2 dt_1 \right] \right\} dt
\end{aligned}
$$

where $K_0 = \gamma(1+C) + i\Delta$, $K_1 = \frac{\sqrt{2\gamma(1+C)}}{K_0}$ and $\Omega_0(t) = \alpha(t)E_0$.

In order to get some intuition about the result, we make the assumption that the amplitude of the control field is constant $E_0(t) = E_0$ and that there is no modulation of the electric field to generate the control field $\alpha(t) = \alpha$. We also make the assumption that the control field is applied long enough to store as much excitations as possible. To make this assumption more precise, when we do the calculation, some decreasing exponential appears like in relation (6). The assumption that we make is to assume that T is several time bigger than the smallest dropping constant $a = \frac{\Delta^2+\gamma^2(1+C)^2}{2\gamma(1+C)\Omega_0^2}$. For instance $T > 3a$. We can now neglect all these exponentials and the solutions can be written as

$$
< S >< S^\star > = \frac{C}{1+C} \left\{ 1 - I_n \frac{2\Omega_0^2\left(\Delta^2+\gamma^2(1+C)^2\right)}{\gamma_N\gamma(1+C)\left(\Delta^2+\gamma^2(1+C)^2\right) + 2\Omega_0^2\gamma^2(1+C)^2} \right\} \tag{22}
$$

$$
< SS^\star > = \frac{C}{1+C} \left\{ 1 - I_n \frac{\Omega_0^2\left(\Delta^2+\gamma^2(1+C)^2\right)}{\gamma_N\gamma(1+C)\left(\Delta^2+\gamma^2(1+C)^2\right) + 2\Omega_0^2\gamma^2(1+C)^2} \right\} \tag{23}
$$

The first thing that we can see is that the error in the number of stored excitations is just twice the error on the spinwave amplitude. So that maximizing one of these quantity makes the otherone maximized too. What we want to see is in which regime the error is the lowest. The error can be written as

$$
Err = \left(\gamma_N \frac{\gamma(1+C)}{\Omega_0^2} + \frac{2\gamma^2(1+C)^2}{\Delta^2+\gamma^2(1+C)^2} \right)^{-1}. \tag{24}
$$

We immediately see that the bigger is the detuning Δ, the bigger is the error.

In the case limit where $\Delta = 0$, the error can be written as

$$
Err = \frac{1}{\gamma_N T_{pulse} + 2}
$$

where

$$
T_{pulse} = \frac{\Delta^2+\gamma^2(1+C)^2}{\gamma(1+C)\Omega_0^2} \tag{25}
$$

is, up to some constant of the order of 1(for instance $\frac{5}{2}$ to have $T = 5a$), the caracteristic time of the probe pulse. Since γ_N is the feedback amplitude, a simple physical interpretation of the error is that the bigger the feedback is compared to the time duration of the pulse, the more the laser have time to average over the noise and so the less is the error.

In the limit $\Delta \to \infty$, the error can be written as

$$Err = \frac{1}{\gamma_N T_{pulse}}.$$

To understand what does mean this limit, let us consider the probe pulse which is given by (21). Using a typical fixed pulse duration T_{pulse}, the phase in the exponential can be written $i\frac{t\Delta}{T_{pulse}\gamma(1+C)}$. Taking the limit $\Delta \to \infty$ corresponds thus to a random AC-stark shift on the probe pulse. And in this limit we see that the condition $\gamma_N T_{pulse} \gg 1$ must be fullfilled to minimize the error as for $\Delta = 0$.

Optimized mode We have computed the efficiency considering a standard mode. Now we consider what happens if we try to optimize the efficiency by changing the probe mode shape. The first step to compute the solution in this section is to compute the optimized mode. This is done by using the method of the lagrange multiplicator assuming the probe pulse is normalized to contain one photon. This method requires to solve the equation

$$\frac{\partial}{\partial \beta}\left(< S > -\lambda \int_0^T < \xi_{in}\xi_{in}^\star >\right) = 0 \tag{26}$$

with the normalization condition

$$\int_0^T < \xi_{in}\xi_{in}^\star >= 1 \Leftrightarrow E_0^2(1+I_N)\int_0^T \beta(t)\beta(t)^\star dt = 1 \tag{27}$$

The expression for $< S >$is given by (2)

$$< S >= \sqrt{\frac{C}{1+C}}\int_0^T < f(t)\xi_{in}(t) > dt = \sqrt{\frac{C}{1+C}}\int_0^T < f(t)E(t) > \beta(t)dt$$

then, using (19), we find

$$< S >= \sqrt{\frac{C}{1+C}}\int_0^T -\alpha^\star(t)\frac{\sqrt{2\gamma(1+C)}}{\gamma(1+C)+i\Delta} < e^{-\frac{h(t,T)}{\gamma(1+C)+i\Delta}}E(t)E(t)^\star > \beta(t)dt \tag{28}$$

In order to simplify the writing of the expression, we introduce $m(t)$ which is defined by

$$m(t) = -\alpha^\star(t)\frac{\sqrt{2\gamma(1+C)}}{\gamma(1+C)+i\Delta}e^{-\frac{h(t,T)}{\gamma(1+C)+i\Delta}}E(t)E(t)^\star \tag{29}$$

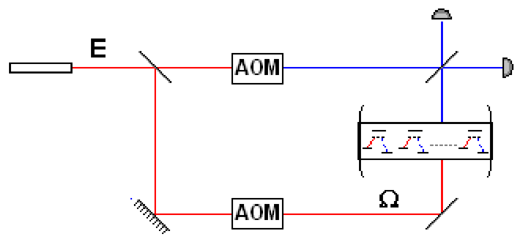

Figure 9: Retrieval scheme

Eq. (28) can then be written as

$$< S >= \sqrt{\frac{C}{1+C}} \int_0^T < m(t) > \beta(t) dt \tag{30}$$

Eq. (26) is then written as

$$\frac{\partial}{\partial \beta} \left(\sqrt{\frac{C}{1+C}} \int_0^T \overline{m(t)} \beta(t) - E_0^2 (1 + I_N) \int_0^T \beta(t) \beta(t)^\star dt \right) = 0 \tag{31}$$

and using the normalization condition (27), it comes

$$\beta(t) = \frac{\overline{m(t)}^\star}{\sqrt{E_0^2(t)(1+I_n)} \sqrt{\int_0^T \overline{m(t')m(t')^\star} dt'}} \tag{32}$$

Now we can compute the efficiency in the same way as we did for the standard mode with the same assumptions. And we obtain the same result to the first order in I_N which is given by the relations (22) and (23). It seems that doing this optimization is useless for low noise. However, to see the effect of the optimization, we will do some comparisons based on simulations. But first we are going to do for retrieval what we have done for storage.

Retrieval The calculation of the retrieval efficiency assuming there is noise in the laser source is very similar to the calculation of the storage efficiency. Considering the experimental setup, we are interested in the overlap of the homodyning pulse and the output pulse.

$$\xi_{homodyning} = \beta_{homodyning}(t) E(t)$$

The output mode is given by the relation (8). As for the storage process, we will consider two different shape for the homodyning pulse. What we are interested in computing are $< A >< A^\star >$ and $< AA^\star >$.

Standard mode The shape of the homodyning pulse is the same as the shape of the output pulse when there is no noise

$$\beta_{homodyning}(t) = -\alpha \frac{\sqrt{2\gamma(1+C)}}{\gamma(1+C)+i\Delta} e^{-\frac{h_0(0,t)}{\gamma(1+C)+i\Delta}}.$$

Then, assuming there is one photon in the homodyning pulse,

$$\xi_{homodyning} = -\Omega(t) \frac{\sqrt{2\gamma(1+C)}}{\gamma(1+C)+i\Delta} e^{-\frac{h_0(0,t)}{\gamma(1+C)+i\Delta}} \sqrt{\frac{1}{1+I_N}} \tag{33}$$

We are now interested in the computation of $< A >$. Using the definition of A in eq. (7) starting with $T_R = 0$, and using the relations (8) and (33), the calculations give the same efficiency as for storage. The integration here is done for time ranging from 0 to infinity. But in practice, it is the same condition as we had for storage : the time duration of the control pulse must be long enough so that all stored excitations are retrieved. Then we can drop all the decreasing exponential that appears during the calculation and we obtain the following results.

$$< A >< A^\star >= \frac{C}{1+C} \left\{ 1 - I_n \frac{2\Omega_0^2 \left(\Delta^2 + \gamma^2(1+C)^2\right)}{\gamma_N \gamma(1+C) \left(\Delta^2 + \gamma^2(1+C)^2\right) + 2\Omega_0^2 \gamma^2(1+C)^2} \right\} \tag{34}$$

$$< AA^\star >= \frac{C}{1+C} \left\{ 1 - I_n \frac{\Omega_0^2 \left(\Delta^2 + \gamma^2(1+C)^2\right)}{\gamma_N \gamma(1+C) \left(\Delta^2 + \gamma^2(1+C)^2\right) + 2\Omega_0^2 \gamma^2(1+C)^2} \right\} \tag{35}$$

Optimized mode The optimization of the homodyning pulse is the same as the optimization of the probe pulse in the storage. And then doing very similar calculations as those that we have done for the storage process, one can obtain the same result for $< A >< A^\star >$ and $< AA^\star >$ that we have just obtained using the standard mode for the homodyning pulse.

5.1.3 Simulations

All the following simulations have been done for storage using MATLAB (Annexe D.1).

Noise generation The first step of the simulation is to generate the noise in the laser beam. We first generate an array that contains white noise. Then the calculation of the noise is done by using relation {15} taking the integral from $t - \frac{5}{\gamma_N}$ to t instead of $-\infty$ to t . We want an array of colored noise that has a size equal to $Nbp \times T$ where Nbp is the number of point per unit of time and T is the duration of the pulse. We thus need an array which has at least a size of $Nbp \times (T + \frac{5}{\gamma_N})$ for the white noise. We then compute the autocorrelation function of the colored noise array. The maximum of the autocorrelation function is equal to the typical intensity of the noise times the integration time $I_{N1}T$ where the intensity I_{N1} depends on the intensity of the white noise. We know need to be able to choose the intensity of the colored noise which means to be able to renormalize the colored noise. Using the property of the autocorrelation function $a^2 < \delta E(t)\delta E(t+\tau) > = < a\delta E(t) a\delta E(t+\tau) >$, we just mutiply the colored noise array by $a = 1/\sqrt{I_{N1}}$ to get an a normilized autocorrelation equal to T, which means that the typical intensity of the colored noise is equal to one. The figure {10} shows the result that we obtain for the averaged noise autocorrelation compared to the theoritical autocorrelation function (16).

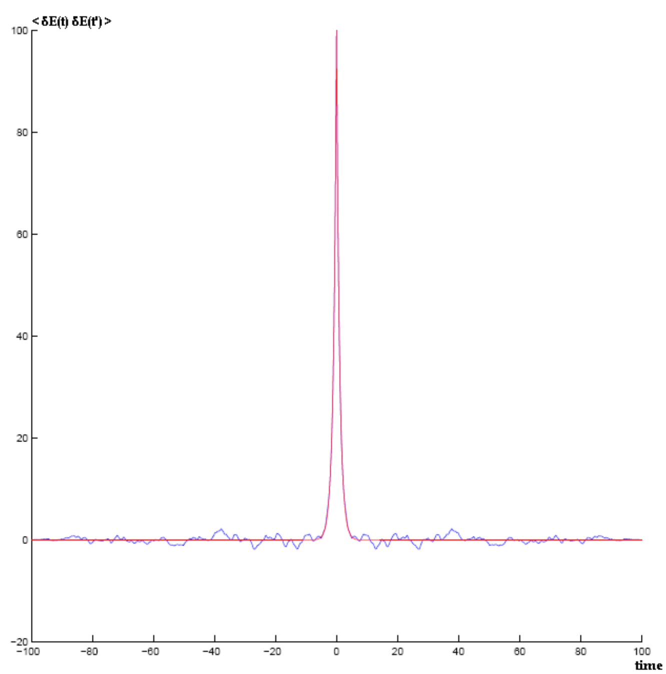

Figure 10: Mean of the autocorrelations functions of 100 noise samples generated (blue) and theoretical autocorrelation function (red), for T=100 with a step time of 0.01, $\gamma_N = 1$ and $E_0 = 1$.

Theoretical results and simulation The figure {11} is a comparison between the theoretical decrease of the efficiency (22) and the simulated efficiency as function of the noise amplitude.

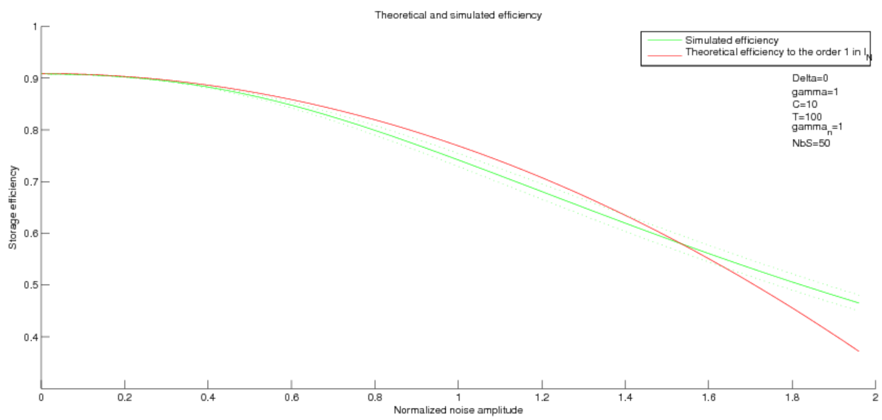

Figure 11: Storage efficiency : simulation and theory with $\Delta = 0$, $\gamma = 1$, $\gamma_N = 1$,$C = 10$, $T = 100$

We are going to describe how this curve has been done. This plot has been computed by considering 50 storage experiments. For each experiment, 100 storage shots are generated for the same noise amplitude. These 100 probe pulses are then normalized to have on average one photon per shot for this experiment.

$$< \int \xi_{in}\xi_{in}^{\star} >= 1 \tag{36}$$

By doing this, we introduce correlation a-posteriori between the probe pulses of the same experiment. That is why we cannot compute the standard error with only one experiment of 100 storage shots. And so the method that we have done is to do several experiments for the same noise amplitude to compute the standard error. However, the efficiency for each noise amplitude has been computed by recycling the same noise samples. That is why the curve is monotonely decreasing without fluctuations of the order of magnitude of the root mean square. However, the noise samples are different for each experiment.

The following plot {12} shows that the theoretical model and the simulation agree for a noise amplitude smaller than 25% of the amplitude of the laser E_0.

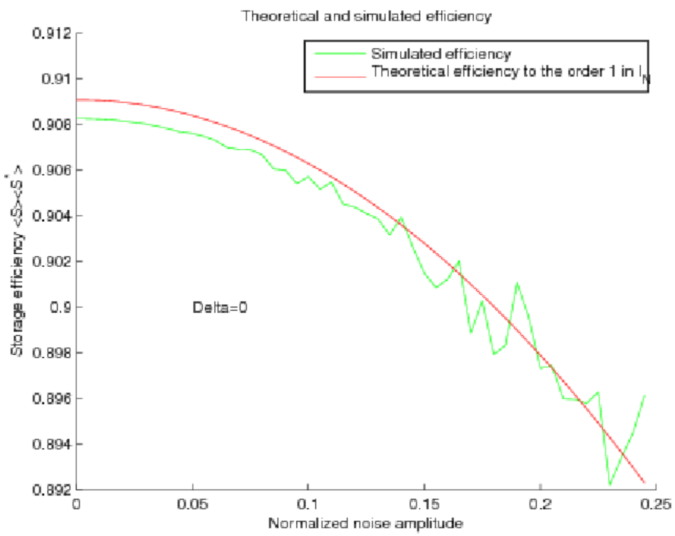

Figure 12: Storage efficiency : simulation and theory (II) with $\Delta = 0$, $\gamma = 1$, $\gamma_N = 1$, $C = 10$, $T = 100$

This plot has been generated by using one experiment of 100 samples of S at each point but without recycling the noise for differents noise amplitude. That is why there are no error bar on this curve. However, one can see that the random fluctuations are around the theoretical plot. The slight offset between the two curves is due to two factors. In the simulation, the pulse energy is finite, T is not infinite so that there is a slight decrease of the efficiency due to the decreasing exponential. The other factor is the numerical error due to the integration with the rectangle method.

These plots show that the efficiency remain still quite high even with big noise amplitude. For instance, assuming a normalized noise amplitude of one, which means there are as many photons that come from noise as from signal, the figure $\{11\}$ shows that the efficiency still remains greater than 70% for an efficiency without noise of 91%.

Simulation of the optimized mode In this section, we have computed an optimized mode to the first order in noise intensity I_N for storage. However, analytically we only had a solution to the first order in the noise intensity and there was no difference between the efficiency computed with the optimized mode and the standard mode. The following plot shows three different efficiciency curves that have been computed using the same noise but with three different probe pulses.

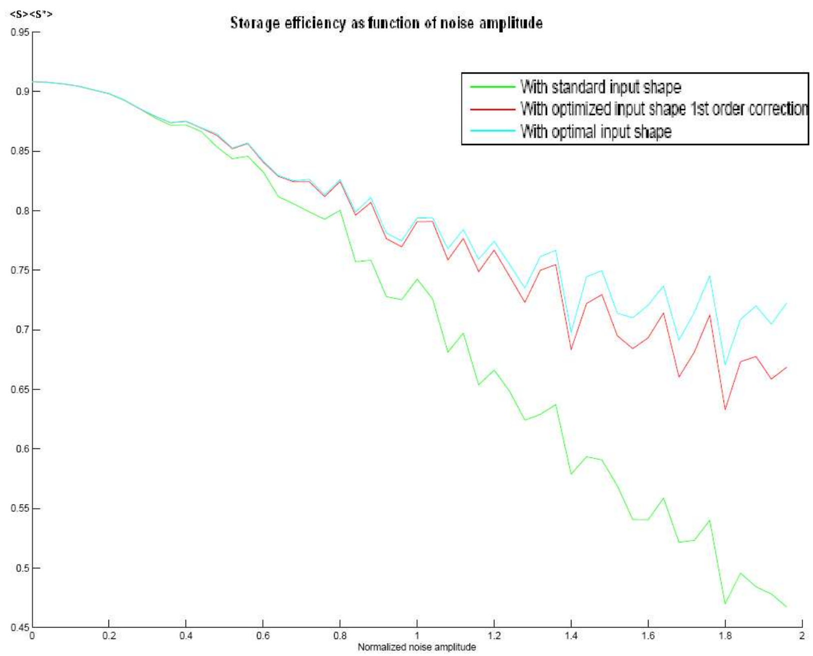

Figure 13: Storage efficiency : optimization

- The green curve has been computed using the standard probe pulse
- The red curve has been computed using the theoretical optimized probe pulse to first order in the noise intensity.
- The blue curve has been computed using a numerical optimized probe.

This last pulse is computed in the following manner

$$\beta_{optimal}(t_i) = \frac{< E(t_i)^\star f(t_i)^\star >}{\sqrt{< E(t_i)E(t_i)^\star >}}$$

which is what is given by the optimal theoretical mode (32) at each time t_i.

As a practical interest, it is impossible to use $\beta_{optimal}$ since it requires to now before the experiment the mean of quantities that are not known. However a practical solution is to measure on a large sample the values of $E(t_i)$ and $f(t_i)$. Then it is possible to compute the mode that would have been optimal to store this sample. Then, assuming the noise has the same properties in the next experiment, one can use the computed optimal mode as probe pulse for the storage in this experiment.

One can see that for these parameters, optimizing the shape of the probe pulse is not really usefull for low noise smaller than 30% of E_0. However, for bigger noise amplitude, it is interesting to consider optimisation.

Influence of the detuning The following plot represents the storage efficiency in function of the detuning for 6 differents values of the noise. For each value of the noise, the calculations have been done both with and without adiabatic elimination of P.

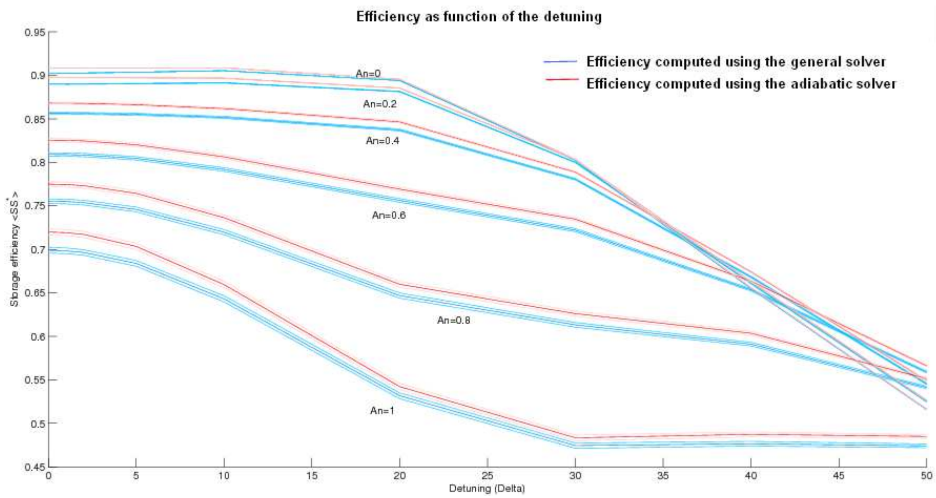

Figure 14: Detuning influence on storage efficiency for differents noise values with $\gamma = 1$, $\gamma_N = 1$, $C = 10$, $T = 100$

Concerning the difference between the two solvers : this plot shows that the solver that computes P without adiabatic elimination adds an offset on the curve but the behavior of the efficiency is preserved. Since the conditions for adiabatic elimination are fullfilled with $TC\gamma = 1000 \gg 1$, this offset is due to the numerical solver.

We see also that as long as we stay in the region where the detuning is small enough to have the approximation $T = \infty$, the bigger is the noise, the worst is the influence of the detuning. This is consistent with the theoretical result. For big detuning, the decay of the efficiency is not only due to the noise but also to the fact that some excitations are not stored (we can not drop the decreasing exponential anymore in the calculations). The crossover between the different curves for different noise might look not really intuitive. However, we remind that this crossover appears in the region where we lack power to store all the pulse. But the bigger is the noise, the bigger is the total power in the control field. And so, at some point, the power given by the noise compensate the loss of efficiency due to the same noise.

The main thing that this plot shows is that the best limit to optimise the efficiency is $\Delta = 0$ which is consistent with the theoretical result.

5.2 Phase noise

All that we have done so far concerned amplitude noise. We are now going to study what happens if there are random fluctuation of the frequency of the laser.

5.2.1 Noise model

Here we assume that the amplitude is constant and equal E_0 and that there is a random fluctation of the frequency $\delta\omega(t)$.

$$E = E_0 e^{i\omega t + i\int_0^t \delta\omega(t_1)dt_1} \tag{37}$$

The model that we assume for $\delta\omega$ is a white gaussian noise with a feedback

$$\dot{\delta\omega}(t) = A_\omega \eta_\omega(t) - \gamma_\omega \delta\omega(t) \tag{38}$$

It is the same equation as the one we used for amplitude noise. And so the autocorrelation function divided by the time duration of the pulse is

$$< \delta\omega(t)\delta\omega(t') >= I_\omega.exp(-\gamma_\omega |t - t'|) \tag{39}$$

with

$$I_\omega = \frac{A_\omega^2}{2\gamma_\omega} \tag{40}$$

5.2.2 Theoretical results

When we have inserted amplitude noise, the Hamiltonian of the system was unchanged. However, when facing frequency noise, one way to deal with it is to go into the interaction picture respect to the Hamiltonian $H_0 = \hbar\delta\omega(t) |e\rangle \langle e|$ .The details are given in the annexe B and it gives the following hamiltonian for the system

$$H = \hbar(\Delta - \delta\omega(t))\sigma_{ee} - (\hbar\Omega(t)\sigma_{es} + \hbar g\xi\sigma_{eg} + h.c) \tag{41}$$

and the master equations of the system becomes

$$\begin{aligned} \xi_{out} &= \xi_{in} + i\sqrt{2\gamma C}P \\ \dot{P} &= -(\gamma(1+C) + i(\Delta - \delta\omega(t)))P + i\Omega S + i\sqrt{2\gamma C}\xi_{in} \\ \dot{S} &= i\Omega^\star P \end{aligned} \tag{42}$$

What we can see at this point for frequency noise is exactly the same as a fluctuation of the detuning. Now we still assume that the adiabatic elimination of P is valid. The exact condition for this adiabatic elimination will be discussed in section 5.2.4. We also assume that Ω_0 does not depends on time.

Storage When considering the storage process and assuming the adiabatic elimination of P, the behaviour of S is described by the two last equations of (42). They can be combined to a single equation of motion :

$$(\gamma(1+C)+i(\Delta-\delta\omega(t))\dot{S}+|\Omega|^2 S=-\Omega^\star\sqrt{2\gamma C}\xi_{in} \tag{43}$$

The method that we chose to solve this equation is to compute an analytical solution for S and then to expand to the lowest order in intensity frequency noise in order to get some simple result. The equation (43) can be written as

$$\dot{S}+a(t)S=b(t) \tag{44}$$

with $a(t)=\frac{|\Omega_0|^2}{\gamma(1+C)+i(\Delta-\delta\omega(t))}$ and $b(t)=\frac{-\Omega_0^\star\xi_{in}\sqrt{2\gamma C}}{\gamma(1+C)+i(\Delta-\delta\omega(t))}$. The solution of this equation is given by the product of two functions $S_0$and S_1. S_0 is the solution of the root equation $\dot{S}+a(t)S=0$. S_1 is the function that we find by replacing S by S_0S_1 in the relation (44). The calculations gives $S_0=Ke^{-\int a(t)}$ and $\dot{S}_1=S_0^{-1}b(t)$ where K is some constant. One can then compute the complete solution

$$S(T)=S_0(T)S_1(T)=e^{-\int_0^T a(t)dt}\int_0^T e^{\int_0^t a(t_1)dt_1}b(t)dt. \tag{45}$$

There is one case that gives some intuition about the influence of the phase noise and that can be solved exactly. Let us assume that the noise $\delta\omega$ is a constant offset ϵ on the frequency rather than a random fluctuation. It corresponds to a feedback $\gamma_\omega=0$ but with a noise intensity I_ω which is still finished. In this case, the solution is easy to find from (45) and using a standard probe pulse ξ_{in} which is given by the relation (5) it follows that

$$SS^\star=\frac{C}{1+C}\frac{1}{1+\epsilon^2/(4\gamma^2(1+C)^2)}\left(1-e^{\frac{-2\Omega_0^2\gamma(1+C)[2\gamma^2(1+C)^2+\Delta(\Delta+\epsilon)+\epsilon^2]}{(\gamma^2(1+C)^2+(\Delta-\epsilon)^2)(\gamma^2(1+C)^2+\Delta^2)}T}\right)^2 \tag{46}$$

Taking the usual limit $T\to\infty$ that corresponds to a time duration of the control field pulse which is large enough to store all the excitations contained in the probe pulse, we can drop the decreasing exponential in {46} and the solution becomes

$$SS^\star=\frac{C}{1+C}\frac{1}{1+\epsilon^2/(4\gamma^2(1+C)^2)} \tag{47}$$

This result shows that the efficiency still does not depend on the detuning when there is an unknown constant frequency shift ϵ on the laser frequency.

This is also a result that can be compared to the solution that we will find in the more general case of a time varying noise frequency. In this case we use the autocorrelation function (39). After computing the Taylor expansion to the first order in I_ω assuming that $\delta\omega\ll\Delta,\gamma(1+C)$, the expression for S is

$$
\begin{aligned}
< S > \quad &= \quad \sqrt{\frac{C}{1+C}} \left[1 - I_\omega \left\{ \frac{1}{4\gamma^2(1+C)^2} \left(1 + \frac{\gamma_\omega(\Delta^2+\gamma^2(1+C)^2)}{\Omega_0^2 2\gamma(1+C)} \right)^{-1} \right.\right. \qquad (48) \\
&+ \quad \left.\left. \frac{\gamma_\omega(\Delta^2+\gamma^2(1+C)^2)}{(\gamma(1+C)+i\Delta)2\gamma(1+C)\left[2\gamma(1+C)\Omega_0^2+\gamma_\omega(\Delta^2+\gamma^2(1+C)^2)\right]} \right\} \right]
\end{aligned}
$$

and it follows that

$$
< S >< S^\star > \frac{C}{1+C} \left[1 - I_\omega \left\{ 1 + \frac{2\Omega_0^2\gamma(1+C) + 4\gamma^2(1+C)^2\gamma_\omega}{4\gamma^2(1+C)^2\left[2\Omega_0^2\gamma(1+C)+\gamma_\omega\left(\Delta^2+\gamma^2(1+C)^2\right)\right]} \right\} \right] = \tag{49}
$$

In order to check if this expression is correct, we compare it to (47). Assuming $\delta\omega(t) = \epsilon$, $< \delta\omega(t)\delta\omega(t') >= \epsilon^2 = I_\omega$. Taking the limit $\gamma_\omega \rightarrow 0$ and assuming that the intensity of the noise is still finished by taking $A_\omega^2 \propto \gamma_\omega$, we find that

$$
< S >< S^\star >= \frac{C}{1+C} \left[1 - I_\omega \frac{1}{4\gamma^2(1+C)^2} \right] \tag{50}
$$

which is exactly the result that we obtain by taking the Taylor expansion of $\{47\}$ to the first order in ϵ^2.

Let us now come back to the description of the average spin wave amplitude squared when there is a time dependent noise (49). The relation (49) describes the behaviour of the amplitude of the spin wave squared as function of the detuning. By looking at this relation, one can say that the bigger is the detuning, the lower is the error due to frequency noise. But one should remember that by increasing the detuning, the power of the laser must be increased to stay in the limit where the whole probe pulse is stored (where we drop the decreasing exponentials). moreover, increasing the detuning will increase the amplitude noise as we have seen before.

However, we get more physical meaning by expressing the power of the control field in function of the typical time duration of the probe pulse (25) $\Omega_0^2 = \frac{\Delta^2+\gamma^2(1+C)^2}{\gamma(1+C)T_{pulse}}$ to transform the relation (49). We then find absorbing a factor of $\frac{1}{2}$ in the definition of T_{pulse}.

$$
< S >< S^\star >= \frac{C}{1+C} \left\{ 1 - I_\omega \left[1 + \left(\frac{\Delta^2+\gamma^2(1+C)^2+4\gamma^2(1+C)^2\gamma_\omega T_{pulse}}{4\gamma^2(1+C)^2(\Delta^2+\gamma^2(1+C)^2)\left[1+\gamma_\omega T_{pulse}\right]} \right) \right] \right\} \tag{51}
$$

We see that the condition on the product $\gamma_\omega T_{pulse}$ to optimize the error is not so easy to get as for the amplitude noise. In the limit $\Delta = 0$, the relation (52) gives

$$
< S >< S^\star >= \frac{C}{1+C} \left\{ 1 - I_\omega \left[1 + \left(\frac{1+4\gamma_\omega T_{pulse}}{4\gamma^2(1+C)^2\left[1+\gamma_\omega T_{pulse}\right]} \right) \right] \right\}.
$$

In this limit, having a time pulse duration larger than the feedback time scale increases the error. And it is still the case as long as the detuning is such that $|\Delta| \leq \sqrt{3}\gamma(1+C)$.

In the limit $\Delta \to \infty$, while T_{pulse} remains finished (which means that we increase also the power of the control field) the relation (52) gives

$$< S >< S^\star >= \frac{C}{1+C}\left\{1 - I_\omega\left[1+\left(\frac{1}{4\gamma^2(1+C)^2\left[1+\gamma_\omega T_{pulse}\right]}\right)\right]\right\}. \quad (52)$$

In this limit $\Delta \gg \gamma(1+C), 2\gamma(1+C)\sqrt{\gamma_\omega T_{pulse}}$, having a short feedback time compared to the pulse duration $\gamma_\omega T_{pulse} \gg 1$ makes the error decreases. It is the same kind of behavior as for amplitude noise.

Retrieval According to our experimental scheme, the retrieval consists in homodyning the output pulse ξ_{out} with the homodyning pulse $\xi_{homodyne}$. The analytical expression for the output pulse ξ_{out} is given by solving the equation system (42) with $\xi_{in} = 0$. It can be done by solving

$$(\gamma(1+C) + i(\Delta - \delta\omega(t))\dot{S} + |\Omega|^2 S = 0$$

which gives $S(T) = Ke^{-\int_0^T \frac{|\Omega_0|^2}{\gamma(1+C)+i(\Delta-\delta\omega(t))}dt}$. Taking $K = 1$ to start the retrieval with one stored excitation $S(0) = 1$, then using the first and the last relations of (42), we find

$$\xi_{out}(T) = \frac{\sqrt{2\gamma C}}{\Omega_0^\star}\int_0^T e^{-\int_0^t \frac{|\Omega_0|^2}{\gamma(1+C)+i(\Delta-\delta\omega(t'))}dt'}dt \quad (53)$$

The analytical expression is the one that we use assuming there is no noise and so the shape is the same as ξ_{out} when there is no noise (8). However taking the fact that we do not compensate the noise on the detuning, we get

$$\xi_{homodyne}(t) = -\sqrt{2\gamma C}\frac{\Omega_0}{\gamma(1+C) + i(\Delta - \delta\omega(t))}e^{-\frac{|\Omega_0|^2 t}{\gamma(1+C)+i\Delta}}. \quad (54)$$

Then, the retrieval efficiency is given by using (7) and it follows that we get the same result as for the storage efficiency

$$< A >< A^\star > \frac{C}{1+C}\left[1 - I_\omega\left\{1 + \frac{2\Omega_0^2\gamma(1+C) + 4\gamma^2(1+C)^2\gamma_\omega}{4\gamma^2(1+C)^2\left[2\Omega_0^2\gamma(1+C) + \gamma_\omega\left(\Delta^2 + \gamma^2(1+C)^2\right)\right]}\right\}\right].$$

5.2.3 Simulation

The figure {15} has been plotted using $\sqrt{\gamma^2(1+C)^2 + \Delta^2} \sim 10$ so that the frequency noise is lower than $\sqrt{\gamma^2(1+C)^2 + \Delta^2}$. The difference between the simulation and the theory is due to the fact that the theoretical result is only a taylor expansion to the second order in frequency noise amplitude.

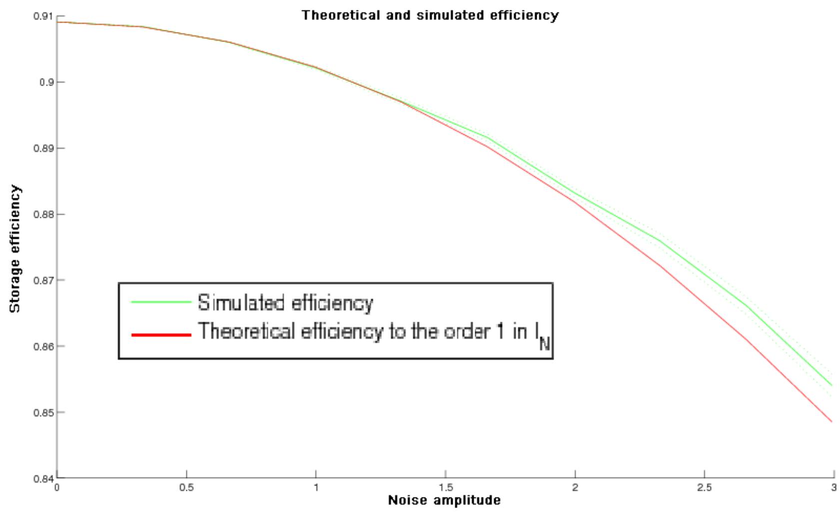

Figure 15: Comparison between simulation and theory for frequency noise with $\Delta = 1$, $\gamma = 1$, $\gamma_\omega = 1$, $C = 10$, $T = 100$

5.2.4 Conditions for adiabatic elimination

The adiabatic elimination of P has been used to compute the theoretical results that we obtained when there is frequency noise. The point is now to give the conditions that the noise must fullfill to make this approximation. The approximation is

$$\dot{P} = 0 \tag{55}$$

When we set the time derivative of P equal to 0 in the second equation of the system (42), we get

$$P = \frac{i\Omega S + i\sqrt{2\gamma C}\xi_{in}}{\gamma(1+C) + i(\Delta - \delta\omega(t))} \tag{56}$$

Taking the derivative of this expression gives

$$\dot{P} = \frac{i(\dot{\Omega} S + \Omega \dot{S})(\gamma(1+C) + i(\Delta - \delta\omega(t))) - i\dot{\delta\omega}(i\Omega S + i\sqrt{2\gamma C}\xi_{in})}{(\gamma(1+C) + i(\Delta - \delta\omega(t)))^2} \tag{57}$$

The condition on $\dot{P}$ that comes from the second equation of the system (42) is

$$\left|\dot{P}\right| \ll \left|-(\gamma(1+C) + \dot{i}(\Delta - \delta\omega(t)))P\right| \tag{58}$$

Here since we are interested in the condition on the noise, the relevant part of (57) is $\frac{\dot{\delta\omega}(i\Omega S + i\sqrt{2\gamma C}\xi_{in})}{(\gamma(1+C)+i(\Delta-\delta\omega(t)))^2}$. The condition (58) then gives $\left|\frac{\dot{\delta\omega}(i\Omega S + i\sqrt{2\gamma C}\xi_{in})}{(\gamma(1+C)+i(\Delta-\delta\omega(t)))^2}\right| \ll$

$\left|(\gamma(1+C)+i(\Delta-\delta\omega(t)))\frac{i\Omega S+i\sqrt{2\gamma C}\xi_{in}}{\gamma(1+C)+i(\Delta-\delta\omega(t))}\right|$. And so it provides some requirement for $\delta\omega$ and $\dot{\delta\omega}$.

$$\left|\dot{\delta\omega}(t)\right| \ll |\gamma(1+C)+i(\Delta-\delta\omega(t))|^2 \tag{59}$$

We then want the noise to fullfill the relation (60). This gives a condition on the average of all correlations to the second order that appears during a pulse of a duration of T.

$$\int_0^T \left\langle \dot{\delta\omega}(t)\dot{\delta\omega}(T)\right\rangle dt \ll \int_0^T \left\langle |\gamma(1+C)+i(\Delta-\delta\omega(t))|^2 \, |\gamma(1+C)+i(\Delta-\delta\omega(T))|^2 \right\rangle \tag{60}$$

We then obtain the following results that limits the feedback γ_ω and the square of the amplitude frequency noise I_ω.

$$\begin{aligned} I_\omega &\ll \left(\gamma^2(1+C)^2+\Delta^2\right)^{\frac{3}{2}} T \\ \gamma_\omega &\ll \sqrt{\gamma^2(1+C)^2+\Delta^2} \end{aligned} \tag{61}$$

If we call the typical frequency shift Γ, $I_\omega = \Gamma^2$ and so the adiabatic elimination of P can be performed as long as the feedback and the frequency shift are small enough (61).

5.3 Time reversal and noise

The expressions of the efficiency that we have found for the storage and for the retrieval processes in presence of noise in the system are both the same. As we will now show, this is a more general property linked to time reversal. Let us call U the unitary matrix that describes the evolution of the system. We now assume that U stands for retrieval. U depends on the control field, starts at a time t_1 and finishes at a time t_2.We write it $U\left[t_2, t_1; \Omega(t)\right]$. Let us say that the system starts in a state $|a\rangle$ which is a spinwave. We call $|b\rangle$ the output mode that we want to retrieve. The probability for the system to go from $|a\rangle$ to $|b\rangle$ is just

$$Eff = |< b|U[T,0;\Omega(t)]|a>|^2 = \left|< a|U^{-1}[T,0;\Omega(t)]|b>\right|^2 \tag{62}$$

where the last part of the relation (62) is due to the unitarity of U.

Eff is just the retrieval efficiency, but the last part of the relation shows that provided we are able to invert the evolution, the storage efficiency is the same. It has been proved in the paper [3] that this can be done by this way

$$U^{-1}[T,0;\Omega(t)] = \hat{\tau}U[T,0;\Omega^\star(T-t)]\hat{\tau} \tag{63}$$

where $\hat{\tau}$ is the time reversal operator. Physically this means that, if we can retrieve the spin wave onto $\xi(t)$ using the control field $\Omega(t)$, we can use the control field $\Omega^\star(T-t)$ to store the incoming pulse $\xi_{in}(t) = \xi^\star(T-t)$.

This works as long as there is no noise in the control field. If there is noise, we should consider the transformation averaged over the noise $< U >$ which is not anymore a unitary transform.

$$< U^{-1}[T,0;\Omega(t)+\delta\Omega(t)] >= \hat{\tau} < U[T,0;\Omega^{\star}(T-t)+\delta\Omega^{\star}(T-t)] > \hat{\tau} \quad (64)$$

However, in practice, we can not time reverse the noise. And what we obtain for U is $U[T,0;\Omega^{\star}(T-t)+\delta\Omega^{\star}(t)]$. In the most general case, the matrix $< U >$ will contain correlation of the noise to all orders $< \delta\Omega(t_1)\delta\Omega(t_2)\delta\Omega(t_3)... >$. Since our model is based on gaussian noise, all correlations with an odd number of terms vanish. We then use the property of gaussian noise that even correlations can always be written as a product of correlation of the 2^{nd} order. For instance, the 4^{th} order can be written

$$\begin{aligned} < \eta(t_1)\eta(t_2)\eta(t_3)\eta(t_4) > &= < \eta(t_1)\eta(t_2) >< \eta(t_3)\eta(t_4) > \\ &+ < \eta(t_1)\eta(t_3) >< \eta(t_2)\eta(t_4) > \\ &+ < \eta(t_1)\eta(t_4) >< \eta(t_2)\eta(t_3) > \end{aligned} \quad (65)$$

Our noise model is not white noise since there is a feedback. Thus, the n-order autocorrelation function is given by

$$\begin{aligned} < \delta E(T_1)\delta E(T_2)...\delta E(T_n) > &= \int_0^{T_1}\int_0^{T_2}...\int_0^{T_n} < \eta(t_1)\eta(t_2)...\eta(t_n) > \\ &\times \; e^{-\gamma_N[(t-t_1)+(t-t_2)+...+(t-t_n)]}dt_1dt_2...dt_n \end{aligned} \quad (66)$$

The odd order autocorrelation functions vanish and since the even autocorrelation function $< \eta(t_1)\eta(t_2)...\eta(t_{2n}) >$ can be developped as a sum of product of autocorrelation functions of the 2^{nd} order, the $< \delta E(T_1)\delta E(T_2)...\delta E(T_{2n}) >$ can be developped as a sum of product of integral of second order white noise correlation functions

$$\begin{aligned} < \delta E(T_1)\delta E(T_2)...\delta E(T_n) > &= \int_0^{T_1}\int_0^{T_2}...\int_0^{T_n} [\\ &+ < \eta(t_1)\eta(t_2) >< \eta(t_3)\eta(t_4) > ... < \eta(t_{2n-1})\eta(t_{2n}) > \\ &+ < \eta(t_1)\eta(t_3) >< \eta(t_2)\eta(t_4) > ... < \eta(t_{2n-2})\eta(t_{2n}) > \\ &\vdots \quad \vdots \\ &+ < \eta(t_{i_1})\eta(t_{i_2}) >< \eta(t_{i_3})\eta(t_{i_4}) > ... < \eta(t_{i_{2n-1}})\eta(t_{i_{2n}}) > \\ &\times \;]\, e^{-\gamma_N[(t-t_1)+(t-t_2)+...+(t-t_{2n})]}dt_1dt_2...dt_{2n}. \end{aligned} \quad (67)$$

and so the even n-order autocorrelation functions for the colored noise can be written as a sum of products of 2^{nd} order autocorrelation functions. Using the property that the 2^{nd} order autocorrelation function only depends on the time difference between the two signals $|t_i - t_j|$ (16), we find

$$< \delta E(t_1)\delta E(t_2)...\delta E(t_n) >=< \delta E(T-t_1)\delta E(T-t_2)...\delta E(T-t_n) > \quad (68)$$

And thus it follows that

$$< U[T, 0; \Omega^\star(T-t) + \delta\Omega^\star(t)] >=< U[T, 0; \Omega^\star(T-t) + \delta\Omega^\star(T-t)] > \quad (69)$$

which means, for the noise model that we have considered, $< U^{-1} >$ can be computed from $< U >$ thanks to the relation (64) and so time reversal still apply.

6 Outlook : Free space

All the work that we have done so far assumes that memory is surrounded by a cavity. An another case to study is the freespace. As for the cavity, the maximal efficiency without noise has already been computed [3]. In the cavity, there is only one mode available for the spinwave and one frequency mode for the probe and output pulses. In the free space regime, the probe pulse is not anymore rigorously monocromatic and there are several spinwaves modes that can be stored. This makes the calculations more complicated although the principle remains the same as for the cavity. Here we will first give the expression of the retrieval and the efficiency when there is no noise. These results comes from the paper [3]. Then we will point out which attempts have been made to get a usefull analytical solution for the efficiency to the first order in noise intensity.

First, here are the results for retrieval when there is no noise. The optimized output mode without noise is

$$\xi_{out}(\tilde{t}) = -\sqrt{d}\tilde{\Omega}(\tilde{t}) \int_0^1 d\tilde{z} \frac{1}{1+i\tilde{\Delta}} e^{-\frac{h(0,\tilde{t})+\tilde{z}d}{1+i\tilde{\Delta}}} I_0\left(2\frac{\sqrt{h(0,\tilde{t})\tilde{z}d}}{1+i\tilde{\Delta}}\right) S(1-\tilde{z}) \quad (70)$$

- S is the spinwave amplitude
- $\tilde{t}$ is the rescaled time $\tilde{t} = t\gamma$
- d is the optical depth of the atomic media which is the equivalent of C for the cavity
- $\tilde{z}$ is a rescaled space variable that allows to integrate all over the atomic ensemble. This is to integrate over all spinwave modes.
- All others values A with a $\tilde{}$ are just rescaled as A/γ.
- $h(0,\tilde{t}) = \int_0^{\tilde{t}} \left|\tilde{\Omega}(t')\right|^2 dt'$
- $I_0(x)$ is the modified Bessel function of the first kind.

The efficiency without noise is given by $\int_0^\infty \xi_{out}(\tilde{t})\xi^\star_{out}(\tilde{t})d\tilde{t}$. Using the following identity [9]

$$\int_0^\infty drre^{-pr^2} I_0(\lambda r) I_0(\mu r) = \frac{1}{2p} e^{\frac{\lambda^2+\mu^2}{4p}} I_0\left(\frac{\lambda\mu}{2p}\right) \quad (71)$$

and assuming $h(0,\infty)$ is big enough, the retrieval efficiency is

$$\eta_r = \int_0^1 d\tilde{z} \int_0^1 d\tilde{z}' k_r(\tilde{z},\tilde{z}') S(1-\tilde{z}) S(1-\tilde{z}') \quad (72)$$

where

$$k_r(\tilde{z},\tilde{z}') = \frac{d}{2} e^{-d\frac{\tilde{z}+\tilde{z}'}{2}} I_0(d\sqrt{\tilde{z}\tilde{z}'}) \tag{73}$$

When introducing noise in the control field $\tilde{\Omega} = \tilde{\Omega_0} + \delta\tilde{\Omega}$, the efficiency is given by $\int_0^\infty \xi_{out}(\tilde{t})\xi^\star_{homodyne}(\tilde{t})d\tilde{t}$, where

$$\xi_{homodyne}(\tilde{t}) = -\frac{1}{\sqrt{1+I_N}}\sqrt{d}\tilde{\Omega}(\tilde{t})\int_0^1 d\tilde{z}\frac{1}{1+i\tilde{\Delta}}e^{-\frac{h(0,\tilde{t})+\tilde{z}d}{1+i\tilde{\Delta}}} I_0\left(2\frac{\sqrt{h_0(0,\tilde{t})\tilde{z}d}}{1+i\tilde{\Delta}}\right)S(1-\tilde{z}). \tag{74}$$

$\xi_{homodyne}$ is generated from the noisy field defined by the Raby frequency Ω. And so $\xi_{homodyne} = \Omega\beta_{shape}K_{norm}$. But the shape β_{shape} of $\xi_{homodyne}$ is computed assuming there is no noise and so it is the same shape as for ξ_{out} when there is no noise. $K_{norm} = 1/\sqrt{1+I_N}$ is a normalization constant ensure that there is one photon in the pulse. $h_0(0,\tilde{t})$ is defined as $h_0(0,\tilde{t}) = \int_0^{\tilde{t}} \left|\tilde{\Omega}_0(t')\right|^2 dt'$.

We will now consider the efficiency. The problem that we are facing is to give an analytical solution for the efficiency assuming Ω_0 is independant of time and that the noise follows the same model as the one we used for the cavity case. The first thing to do is to expand the Bessel function that contains noise $I_0\left(2\frac{\sqrt{h(0,\tilde{t})\tilde{z}d}}{1+i\tilde{\Delta}}\right)$. This expansion depends on $I_1\left(2\frac{\sqrt{h_0(0,\tilde{t})\tilde{z}d}}{1+i\tilde{\Delta}}\right)$. The main problem that we then face is to compute this kind of integrals :

$$\int_0^\infty drr^\alpha e^{-pr^2} I_0(\lambda r)I_1(\mu r) \tag{75}$$

Facing this problem, one can think about using a generalized form of the solution which is used when there is no noise (71). This is the following one [9]:

$$\begin{aligned}\int_0^\infty drr^\alpha e^{-pr^2} I_\mu(br)I_\nu(cr) &= \frac{b^\mu c^\nu p^{(-\frac{(\alpha+\mu+\nu)}{2})}}{(2^{(\mu+\nu+1)}\Gamma(\nu+1))} \\ &\times \sum_{k=0}^\infty \frac{\Gamma\left(k+\frac{(\alpha+\mu+\nu)}{2}\right)\left(\frac{b^2}{(4p)}\right)^k {}_2F_1(-k,-\mu-k;\nu+1;\frac{c^2}{b^2})}{(\Gamma(\mu+k+1)k!)}\end{aligned} \tag{76}$$

where ${}_2F_1(a,b;c;z)$ is Gauss's hypergeometric function and Γ is the generalized factorial function. However, this was hard to make simplifications on series terms in order to make some well known functions merge in the result.

One other method to compute an approximative analytical result of (75) is to approximate the Bessel functions by their asymptotic forms $\hat{I}_0$ and $\hat{I}_1$. The two functions $\hat{I}_0(x)$ and $\hat{I}_1(x)$ are discontinuous since they are defined by [10]

$\hat{I}_0(x) = 1$ for small x ($x \ll 1$) and $\hat{I}_0(x) = \frac{1}{\sqrt{2\pi x}}e^x$ for big x, ($x \gg \frac{1}{4}$)

$\hat{I}_1(x) = \frac{x}{4}$ for small x ($x \ll \sqrt{2}$) and $\hat{I}_1(x) = \frac{1}{\sqrt{2\pi x}}e^x$ for big x ($x \gg \frac{3}{4}$)

This requires also to choose the cross-over between big x and small x which can be chosen to be $x_0 \approx 0.7$ for both functions. However, this method still gives an integral which is hard to compute analytically.

7 Conclusion

We have seen that quantum memories are required to enhance quantum communications distance. And so the storage and retrieval efficiency of the photons is the criteria that is used to quantify the efficiency of the memory. In practice there is always some noise in the electromagnetic fields which are used. We have considered one kind of experimental setup and, under the assumptions that $TC\gamma \gg 1$ and that the memory was surrounded by a cavity, we have derived the analytical solutions for the storage and retrieval efficiency for low noise considering two differents types of noise. We have seen that the efficiency still remain high compared to the efficiency without noise. Moreover, for amplitude noise, it is possible to shape the probe pulse to optimize the efficiency taking into account the noise. One interesting result is also the fact that the noise model we have considered does not affect the time reversal property. And so, all its usefull applications which are pointed out in this paper [3] remains still valid.

The natural next step to this work would be to derive analytical solutions or to perform simulations in the case of free space. Some attempts have been pointed out in the outlook section.

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A Rabi oscillations

Let us consider a two states system $|e\rangle$ and $|g\rangle$. This is an atom with a ground state $|g\rangle$ and an excited state $|e\rangle$. We apply an electromagnetic field of frequency $\omega_0 = \omega_{eg} - \Delta$ to this system where the detuning Δ is the difference between the transition frequency and the field frequency. Then, the atom will jump from one state to the other with a fixed frequency. More accurately, if the atom is in the ground state at the beginning, its probability $P_e(t)$ to be measured in the excited state increases until it reaches a maximum (equal to one if $\Delta = 0$). Then the probability to be in the ground state increase until it reach the probability of one. And it starts again. The frequency at which this happens is called the Rabi frequency and it can be derived by considering the Hamiltonian of the system

$$\hat{H} = \hbar\omega_{eg} |e\rangle \langle e| - \hat{d}.E_0 \cos(\omega_0 t) \tag{77}$$

and the state vector of the system

$$|\Psi(t)\rangle = C_g(t) |g\rangle + C_e(t) e^{-i\omega_{eg} t} |e\rangle \tag{78}$$

Then using the Schrödinger equation that describes the time evolution of the system $i\frac{\partial|\Psi(t)\rangle}{\partial t} = \hat{H} |\Psi(t)\rangle$, assuming that the atom is in the ground state at the beginning $C_e(0) = 0$ and $C_g(0) = 1$, using the Rotating Wave Approximation to keep only low frequency terms in $\omega_{eg} - \omega_0$ and dropping the terms in $\omega_{eg} + \omega_0$, the following results are obtained

$$P_e(t) = \frac{(d^\star_{eg} E_0)^2}{\Omega^2 \hbar^2} \sin^2(\frac{\Omega t}{2}) \tag{79}$$

$$\Omega = \sqrt{\left(\Delta^2 + \frac{\left(d^\star_{eg} E_0\right)^2}{\hbar^2}\right)} \tag{80}$$

In this report, the resonant Rabi frequency describes the electric field and is defined as the usual resonant Rabi frequency divided by 2.

$$\Omega_0 = \frac{d^\star_{eg} E_0}{2\hbar} \tag{81}$$

where $d^\star_{eg}$ is the dipole matrix element $d^\star_{eg} =< e|\widehat{d}|g >$.

B Derivation of the Hamiltonian of the system including phase noise

The derivation of the hamiltonian is almost the same as in the paper [2]. The electric field operator for the quantized field of the cavity is given by

$$\hat{E}_1(z,t) = \epsilon_1 \left(\frac{\hbar\omega_1}{2\epsilon_0 V}\right)^{\frac{1}{2}} \left(\hat{a}e^{i(\omega_1 \frac{z}{c}+\phi(t))} + \hat{a}^\dagger e^{-i(\omega_1 \frac{z}{c}+\phi(t))}\right) \tag{82}$$

where $\hat{a}^\dagger$ is the mode creation operator, ω_1 is the mode frequency, ϵ_1 is the polarization unit vector, ϵ_0 is the permittivity of the media in the cavity (vacuum or air), and V is the quantization volume for the field. The excitations are created in the cavity thanks to a laser. Since the laser has frequency noise, we add the frequency noise of the excitations in the hamiltonian of the quantized electric field and so $\phi(t)$ is the phase noise that results from the frequency noise $\delta\omega(t)$.

The single-mode plane-wave control field with frequency ω_2 is described by an electric field vector

$$E_2(z,t) = \epsilon_2 \xi_2(t) cos(\omega_2(t - \frac{z}{c}) + \phi(t)) \tag{83}$$

where ϵ_2 is the polarization unit vector, c is the speed of light and $\xi_2(t)$ is the amplitude. Since the control field is derived from the same laser as the one which is used to generate the probe pulse, the frequency noise is described by $\phi(t)$. Then using the dipole approximation, the Hamiltonian is

$$\hat{H} = \hat{H}_0 + \hat{V} \tag{84}$$

where $\hat{H}_0 = \hat{H}_{field} + \hat{H}_{atoms}$ is the Hamiltonian of the system without any interaction and $\hat{V}$ describes the coupling between the atoms and the free quantized electromagnetic field

$$\hat{H}_{field} = \hbar\omega_1 \hat{a}^\dagger \hat{a} \tag{85}$$

$$\hat{H}_{atoms} = \sum_{i=1}^{N} \left(\hbar\omega_{sg}\hat{\sigma}^i_{ss} + \hbar\omega_{ge}\hat{\sigma}^i_{ee}\right) \tag{86}$$

$$\hat{V} = -\hat{d}.\left(E_2(z,t) + \hat{E}_1(z)\right) \tag{87}$$

where $\hat{d} = \vec{d}_{eg} \left|e\right\rangle \left\langle g\right| + \vec{d^\star}_{eg} \left|g\right\rangle \left\langle e\right|$ is the dipole moment operator for one atom.

In order to give an expression of $\hat{V}$, we need to use the rotating wave approximation (Annexe C.2). This approximation applies when very high frequency terms of same amplitude as lower frequency terms appear while going into the rotating frame. The rotating frame (Annexe C.1) is the frame in which the system's evolution is taken with respect to the quantized electromagnetic field frequency. In this case we drop the high frequency terms because they quickly average to 0 and so the dynamic of the system is driven by the low frequency terms.

$$\hat{V} = -\hbar \sum_{i=1}^{N} \left(\Omega(t) \hat{\sigma}_{es}^{i} e^{-i\omega_2 (t - z_i/c)} e^{-i\phi(t)} + \hat{a} g e^{i\omega_1 z_i/c} \hat{\sigma}_{eg}^{i} e^{-i\phi(t)} \right) + h.c \tag{88}$$

where $h.c$ stands for Hermitian conjugate, z_i is the position of i^{th} atom, $\hat{\sigma}_{\mu\nu}^{i} = |\mu\rangle^{i} \langle\nu|^{i}$ is the transition operator for i^{th} atom between states μ and ν, $\Omega(t) = \frac{\langle e|\hat{d}.\epsilon_2|s\rangle}{2\hbar} \xi_2(t)$ is the Rabi frequency (Annexe A) of the classical control field, and $g = \langle e| \hat{d}.\epsilon_1 |s\rangle \sqrt{\frac{\omega_1}{2\hbar\epsilon_0 V}}$ is the coupling constant between the atoms and the quantized electromagnetic field mode.

We then introduce the collective operators

$$\hat{\sigma}_{\mu\mu} = \sum_{i=1}^{N} \hat{\sigma}_{\mu\mu}^{i} \tag{89}$$

$$\hat{\sigma}_{es} = \sum_{i=1}^{N} \hat{\sigma}_{es}^{i} e^{-i\omega_2 (t - \frac{z_i}{c})} \tag{90}$$

$$\hat{\sigma}_{eg} = \sum_{i=1}^{N} \hat{\sigma}_{eg}^{i} e^{-i\omega_1 (t - \frac{z_i}{c})} \tag{91}$$

$$\hat{\sigma}_{sg} = \sum_{i=1}^{N} \hat{\sigma}_{sg}^{i} e^{-i(\omega_1 - \omega_2)(t - \frac{z_i}{c})} \tag{92}$$

$$\xi = \hat{a} e^{i\omega_1 t} \tag{93}$$

It then follows that $\hat{H}$ is written $\hat{H}_0 - \hbar \left(\Omega(t) \hat{\sigma}_{es} e^{-i\phi(t)} + \xi g \hat{\sigma}_{eg} e^{-i\phi(t)} \right) + h.c$

Then going into the rotating frame respect to $\hat{H}_1 = \hbar(\omega_1 + \delta\omega(t))\hat{\sigma}_{ee}$ yields to the interaction Hamiltonian

$$\hat{H}_I(t) = \hat{U}^{-1}(t)\hat{H}(t)\hat{U}(t) - \hat{H}_1(t) \tag{94}$$

where

$$\hat{U}(t) = e^{-\int_0^t \frac{i}{\hbar} \hat{H}_1(t')dt'} = 1 + (e^{-i(\omega_1 t + \phi(t))} - 1)\hat{\sigma}_{ee} \tag{95}$$

since $\int_0^t \delta\omega(t')dt' = \phi(t)$.

$\hat{H}_1$ is chosen to have an Hamiltonian that describes the interaction of the system with an electromagnetic field with a frequency equal to $\omega_1 + \delta\omega(t)$.

The Hamiltonian in the rotating frame that is used to compute the efficiency in presence of frequency noise is given by

$$\hat{H}_I(t) = \hbar(\Delta - \delta\omega(t))\sigma_{ee} - (\hbar\Omega(t)\sigma_{es} + \hbar g \xi \sigma_{eg} + h.c) \tag{96}$$

The spin wave operator $\hat{S} = \frac{\hat{\sigma}_{gs}}{\sqrt{N}}$ and the polarization operator $\hat{P} = \frac{\hat{\sigma}_{ge}}{\sqrt{N}}$ are defined in the same manner as when there is no noise. Their equations of motion are derived from the Hamiltonian (96) and taking into account the decay γ.

C Physical tools

C.1 Rotating frame

Starting with a Hamiltonian in the Schrödinger picture, the evolution of the system is describe by

$$i\hbar \frac{\partial \left|\Psi(t)\right\rangle}{\partial t} = \hat{H}\left|\Psi(t)\right\rangle \tag{97}$$

Going into a given rotating frame means that we choose an operator $\hat{H}_0(t)$ and we transform $\left|\Psi(t)\right\rangle$ and $\hat{H}$ into $\left|\tilde{\Psi}(t)\right\rangle$ and $\tilde{\hat{H}}$ which are hopefully much simpler provided we have chosen a judicious $\hat{H}_0(t)$.

Let us consider the unitary operator $\hat{U}(t) = e^{-\frac{i}{\hbar}\int_0^t \hat{H}_0(t')dt'}$. We unitary transform $\left|\Psi(t)\right\rangle$ to $\left|\tilde{\Psi}(t)\right\rangle$ using

$$\left|\tilde{\Psi}(t)\right\rangle = U^{\dagger}\left|\Psi(t)\right\rangle \tag{98}$$

We now then write the corresponding Hamiltonian

$$\begin{aligned}
i\hbar \frac{\partial \left|\tilde{\Psi}(t)\right\rangle}{\partial t} &= i\hbar \frac{\partial \left|\hat{U}(t)^{\dagger}\Psi(t)\right\rangle}{\partial t} \\
&= i\hbar \hat{U}^{\dagger}(t)\frac{\partial \left|\Psi(t)\right\rangle}{\partial t} + i\hbar \frac{\partial \hat{U}(t)^{\dagger}}{\partial t}\left|\Psi(t)\right\rangle \\
&= \hat{U}^{\dagger}(t)\hat{H}\left|\Psi(t)\right\rangle + i\hbar \frac{\partial \hat{U}(t)^{\dagger}}{\partial t}\left|\Psi(t)\right\rangle \\
&= \hat{U}^{\dagger}(t)\hat{H}\left|\Psi(t)\right\rangle - \hat{H}_0(t)\hat{U}^{\dagger}\left|\Psi(t)\right\rangle \\
&= \hat{U}^{\dagger}(t)\hat{H}\hat{U}\left|\tilde{\Psi}(t)\right\rangle - \hat{H}_0(t)\left|\tilde{\Psi}(t)\right\rangle
\end{aligned}$$

and so

$$i\hbar \frac{\partial \left|\tilde{\Psi}(t)\right\rangle}{\partial t} = \tilde{\hat{H}}\left|\tilde{\Psi}(t)\right\rangle \tag{99}$$

with

$$\tilde{\hat{H}}(t) = \hat{U}^{\dagger}(t)\hat{H}(t)\hat{U}(t) - \hat{H}_0(t) \tag{100}$$

$\tilde{\hat{H}}(t)$ is the Hamiltonian in the rotating frame

C.2 Rotating Wave Approximation

The Rotating Wave Approximation consists in keeping only low frequency terms in an Hamiltonian. In order to illustrate how this approximation can be performed, we consider the two level atom system that interact with an electromagnetic field. This is the same we used to define the Rabi frequency.

$$\hat{H} = \hbar\omega_{eg} \left|e\right\rangle \left\langle e\right| - \hat{d}.E_0 \cos(\omega_0 t) \tag{101}$$

Going into the interaction picture which means going into the rotating frame respect to $\hat{H}_0 = \hbar\omega_{eg} \left|e\right\rangle \left\langle e\right|$, the interaction Hamiltonian is given by

$$\begin{aligned} \hat{H}_I &= \hat{U}^\dagger(t)\hat{H}(t)\hat{U}(t) - \hat{H}_0(t) \\ &= -\hbar\Omega(e^{-i(\omega_{eg}-\omega_0)t} + e^{i(\omega_{eg}+\omega_0)t}) \left|e\right\rangle \left\langle g\right| - \hbar\Omega^\star(e^{-i(\omega_{eg}+\omega_0)t} + e^{i(\omega_{eg}-\omega_0)t}) \left|g\right\rangle \left\langle e\right| \end{aligned} \tag{102}$$

Then assuming that $\Delta = \omega_{eg} - \omega_0 \ll \omega_{eg} + \omega_0$, the high frequency terms are dropped since they rapidely oscillate and averages to 0 quickly compared to the dynamic of the system which is driven by the low frequency terms. And so the interaction Hamiltonian becomes

$$\hat{H}_I = -\hbar\Omega e^{-i(\omega_{eg}-\omega_0)t} \left|e\right\rangle \left\langle g\right| - \hbar\Omega^\star e^{i(\omega_{eg}-\omega_0)t} \left|g\right\rangle \left\langle e\right| \tag{103}$$

One can wonder why we switched to the interaction picture. This is due to the fact that the state vector of the system is given by

$$\left|\Psi(t)\right\rangle = C_g(t) \left|g\right\rangle + C_e(t) e^{-i\omega_{eg}t} \left|e\right\rangle \tag{104}$$

and so going to the interaction picture just moves the oscillations of the excited states from the state vector to the interaction Hamiltonian.

D Informatic tools

In order to compute the simulation and to get the analytical expressions for the efficiency, we respectively used MATLAB and MAPLE.

D.1 MATLAB simulations

The MATLAB simulations have been performed using different m-files. The architecture is very simple.

- One m-file is used to generate some noise samples which are stored on a remote scratch disk.
- To compute the efficiency, the noise is loaded, renormalized to get the proper amplitude and the efficiency is computed using a chosen solver.
- Then the last step is the plot using MATLAB plotting tools.

The different types of solver are each in one m-file. The RK4solver uses the Runge Kutta method of the 4^{th} order to compute the storage efficiency without performing the adiabatic elimination. The adiabatic solver uses the adiabatic elimination and thus does not require any special numerical tool since we know the analytical solution. However it requires to compute integrals with fixed step time. For instance it is not possible to use optimal method with error control and adaptive step integration since there is noise in the signal that we consider.

D.2 MAPLE calculations

There is no general method to get the result you want using MAPLE. However here are a few thing that have been useful to get the solutions.

- To group as much as possible all constants greatly improve the quality of the result and the time it takes to compute the result.
- Using commands like *convert* to transform big additions in a list makes calculations easier since it enables to focus independently on each part of the expression.
- Using the *subs* and *algsubs* commands was helpful to replace correlations of the noise by analytical expressions of these correlations
- Doing an expansion of an expression $\eta(t)$ that depends on time can be perform quite easily by multiplying $\eta(t)$ by x and then using a Taylor expansion of the expression respect to x. This is especially really helpful if there are expressions where each η are not taken at the same time such like $\eta(t)e^{\int_0^t \eta(t_1)dt_1}$.

- Sometimes, MAPLE does not compute an integral even if you see that it is the kind of integral you could solve with an easy method by hand (but that could take plenty of time). In this case, what may be tried is to group constants that can be grouped and if this is can be written in term of a sum, to integrate each part of the sum step by step.