
Three-Dimensional Theory of Light-Matter Interactions

*Faculty of Science
University of Copenhagen
The Niels Bohr Institute*

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Martin Westring Sørensen

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Abstract

In this thesis we lay the foundations for a three dimensional theory describing light interacting with an atomic ensemble. The thesis is divided in four parts. In Part I we introduce quantum optics in relation to the field of quantum computation. We present leading results on light interacting with atomic ensembles, and discuss the problems associated with these results. In Part II we derive three dimensional equations of motion describing light interacting with atomic ensembles. In deriving these results we first consider a classical Lagrangian, from which we may derive classical equations of motion. Following conventional quantization procedure we derive a quantized Hamiltonian, and using Heisenberg equation of motion we derive our three dimensional equations of motion. We will also examine the detailed structure of the light field entering the equations. The equations we will cast in a form ideally suited perturbative treatment. In the final part of Part II we derive a general solution to the equations of motion. This includes detailed knowledge of initial- as well as boundary condition effects in the general solution. In Part III we solve the equations by using Born approximations. We will focus on the equation describing the atomic behavior, and show that it is hermitian, i.e. physically meaningful. Finally we consider the atomic equation in single mode, and derive nontrivial terms describing decay, and *light induced interactions* between the atoms. In Part IV we conclude the derivations, and comment on future prospects.

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Martin Westring Sørensen
westring@nbi.dk

Notations and Conventions

The use of fonts

Vector fields and vectors are written in bold face:

$$\hat{\mathbf{D}}(\mathbf{r}, t), \hat{\mathbf{A}}(\mathbf{r}, t), \hat{\mathbf{\Pi}}(\mathbf{r}, t), \chi(\mathbf{r}, t), \dots \mathbf{r}, \mathbf{k}, \dots$$

Scalar fields and scalars are written in normal font:

$$\hat{\phi}(\mathbf{r}, t), \epsilon(\mathbf{r}), \omega_L, k_L, \dots$$

The use of indices

The italic i is the imaginary unit, i.e., $i = \sqrt{-1}$

$$j \quad \text{typically labels atoms, } \sum_j^{\text{Atoms}}.$$

l, m, n labels spatial vector components, in some cases, we will also use: k, r, s, t .

Greek letters as index are only used when considering four vector components.

Matrices and vector juxtapositions, when not considered component-wise, are marked by a double bar:

$$\bar{\bar{a}}^j, \bar{\bar{l}}^j, \bar{\bar{G}}^{(\pm)}(\mathbf{r}, t | \mathbf{r}', t')$$

Differential Operators

We generally write out differentiations, however when notation is cumbersome we use the following shorthand notation:

$$\frac{\partial}{\partial t} \mathbf{A}(\mathbf{r}, t) = \dot{\mathbf{A}}(\mathbf{r}, t).$$

When writing the differential vector ∇ we mean:

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

In this notation the divergence of a vector field $\mathbf{A}(\mathbf{r}, t)$ is given as:

$$\nabla \cdot \mathbf{A}(\mathbf{r}, t) = \frac{\partial A_x(\mathbf{r}, t)}{\partial x} + \frac{\partial A_y(\mathbf{r}, t)}{\partial y} + \frac{\partial A_z(\mathbf{r}, t)}{\partial z}$$

Einstein Summation Notation

We generally use vector symbols as \cdot and \times , however when explicitly writing out component manipulations we use the summation convention: Whenever an index occurs two times in a product it is to be summed. This means:

$$\mathbf{A} \cdot \mathbf{B} = A_n B_n = \sum_{n=1}^3 A_n B_n$$
$$(\mathbf{A} \times \mathbf{B})_n = \varepsilon_{nml} A_m B_l = \sum_{ml=1}^3 \varepsilon_{nml} A_m B_l.$$

Where the antisymmetric tensor ε_{lmn} is defined such that:

$$\varepsilon_{123} = +1$$

Fourier transform

In Fourier transform we choose the convention, where the factor of 2π always appear with the momentum integral. For example in three dimensions this reads,

$$f(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{r}} \tilde{f}(\mathbf{k})$$
$$\tilde{f}(\mathbf{k}) = \int d^3r e^{i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{r})$$

Part I

Motivation

Introduction

In 1913, when Niels Bohr wrote the revolutionary paper on the hydrogen atom, the first steps in what was to become the most successful theory on the small things in this universe took place. In the period until 1925 this theory grew to its final form, today known as Quantum Mechanics (QM). The theory deals with things so small that they are beyond the reach of our daily intuition, the theory therefore relies mostly on mathematical reasoning. Only few people working with the theory of QM do not marvel at its beauty, are not stunned by its predictive powers. To the extent on which we have been capable of experimentally testing the theory, it has never been wrong.

The formalism of QM is difficult to understand and interpret, even, or especially, on a philosophical level. Many have seen the seemingly paradoxical nature of QM, and have had to choose a point of view on the matter. One of the difficult concepts in QM is the difference between a system freely evolving and the measurement on this system. Schroedinger's cat experiment is one such example. The difference between system and measurements on the system was hard to accept for the scientist Albert Einstein, which led him, in collaboration with B. Podolsky and N. Rosen, to formulate the EPR-paradox(1). This states that if QM is correct it will also have to be unlocalized. Suppose one has some knowledge of a combined system composed of two systems. If one separates the two systems, it should be possible by making a measurement on one system, to state something about the other system. Then one could make a measurement on the other system thus acquiring additional information on the system. Suppose the two measurements corresponded to two in QM non-commuting operators, a violation of the founding principle of QM would have taken place. This would happen unless a measurement on one of the system does effect the outcome of a measurement on the other. Two systems subject to this behavior are said

to be entangled or inseparable. For many years this discussion of entanglement was thought to be somewhat subtle, until J.S. Bell (2), in a simple inequality, outlined the difference between QM and a local theory. Outcome of experimental test of this inequality has voted for QM(3). The group of Greenberger Horne and Zeilinger have cast the problem concerning a local theory vs. QM in a form involving no inequalities.* They showed by considering certain three particle correlation experiment that the result of local theory and quantum mechanics are in complete contradiction(5). Again the outcome of experimental tests has voted for QM(6). Yet others have in this respect tested QM(7; 8; 9), and it is considered a fact that physical systems possess on a small scale properties such as *Entanglement*.

1.2 Quantum Information

Having the validity of QM tested, one may naturally discuss the prospects of QM. The peculiarities of the theory has over the past decade found technological use, and has opened up an new research area, Quantum Information. (QI) This process has been facilitated by the increasingly more refined experimental techniques, where single atomic systems can be addressed. Implementations of quantum effects are within our grasp, the quantum computer is no longer utopia.† A quantum computer is a computer capable of manipulating quantum objects, e.g., a superposition of two states. Such a machine could solve problems exponentially faster than any classical computer(10), in particular prime factorization of large integers(11). Most of today's encryption protocols are based on the abilities of classical computers, or say lacking ability to prime factorize, hence a quantum computer would revolutionize this field of research. As quantum theory provides a problem regarding cryptography, it gives a solution as well. Using the principles of entanglement it has been shown possible to devise completely secure information channels(12; 13). Even quantum error correcting codes has been shown to exist(14).

The invention of quantum computers has opened up a new goal for quantum information theory, "...namely, to understand the kind and quality of channel resources, needed for the transmission of intact quantum states,..." Charles H. Bennett *et al.*(15). Once we want to implement quantum computers we need to be able to transfer quantum data. The key ingredients in a quantum computer is the ability to process and transfer quantum data. Regarding transferring quantum data, one would think of a photon in an optic fiber, however the probability of losing the photon depends exponentially on length. We thus stand with little chance of transferring quantum data over long distances. Methods has been devised where noisy channels may be used as quantum state carriers.

*See, e.g., the book (4) pg 507 ff.

†This might be a reflection of the authors personal opinion and belief.

The idea is to use entangled pairs of particles as a quantum information channel. Implementing quantum computers as intermediate steps along the channel, a large set of badly entangled quantum bits may be purified, and an arbitrary good set of entangled states acquired(16; 17; 15; 18). This intermediate insert of quantum computers is referred to as quantum repeaters. This whole apparatus, novel, powerful and complete as it may seem, is feasible only to the extent which we may gain access to quantum-like systems.

The most natural construction of a quantum system is a single atom accessed by a single photon. The photon being the quantum information carrier, and the atom the information storage. Proposals as well as experiments has suggested the use of single atoms, ions, caught in a cavity(19; 20; 21). The atoms may then be addressed by an interacting light field. The problems concerning this type of single atom systems, are the need for strong coupling between photons and atoms, decoherence, and loss: “unfortunately ...*the experimental progress has been hampered by decoherence caused by cavity decay and spontaneous emission from the atoms.*”, Anders Sorensen *et al.* (22). It is important that the photon interacts with the atom before the photon is lost. Naturally one could suggest, that if instead of one atom, with which the photon could interact positively, we had two atoms, would that not improve the rate of success? Proposals has been made in which a macroscopic ensemble of atoms and a laser light pulse make up an accessible quantum system. Using an atomic ensemble it is shown possible to encode the information of a light pulse completely.(23) In the article (24) by Lene Hau *et al.* experimental realizations of light speed reduction to 17m per second is being reported, in later work, even coherent trapping of light in the gas have been experimentally verified. There is reason to believe that light pulses and ensembles constitute an accessible quantum system. Many theoretical proposals devise methods for experimentally realizing quantum mechanical properties such as entanglement in frames of light pulses and ensembles(25; 26; 27; 28). The possibility of gaining entanglement between macroscopic atomic ensembles ultimately connects this research to the greater task of constructing a quantum computer. The group, Eugene Polzik *et al.*, reports on an experimental realization of macroscopic entanglement.(29; 30; 31) The research field regarding entanglement of macroscopic ensembles, has thus reached a level of achievement, of which one may ask, “how much further can we go?” The simple quantum system of one atom in a cavity, met the problem of loss. To what extend are the problems concerning ten to the tenths of atoms accounted for?

The present work on light interacting with ensembles is based essentially on a one-dimensional treatment of the system, brought forward by using a far detuned laser.(32) In this respect the governing dynamics are controlled by a simple linear Hamiltonian $\alpha \hat{S}_z \hat{J}_z$. In the article (33), the group examines the limitations regarding inherent noise in the system, evolving according to the simple linear Hamiltonian. The conclusions drawn on that account is, that shot-noise from the light

is being projected onto the atoms, which is then back acting onto the light. The shot noise limit is a fundamental limit in quantum mechanics, and as suggested in the paper, a way to circumvent this problem is to use squeezed light, though this meets limitations as well. For a more dense sample of atoms, e.g. a Bose condensate, diffraction effects, not dealt with in the simple theory comes into play. In (34), the group J.H. Müller *et al.* describes a semi-classical treatment of light interacting with dense ensemble of atoms. Semi-classical in the respect that diffraction effects are treated classically, whereas light-atom interactions are treated with the usual simple Hamiltonian. If we want to know in detail the limitations of entanglement between atomic ensembles we still need the proper description. *“Ultimately, a proper quantum model will have to take into account the scattering induced dynamics of the density correlation function of the sample, which determines the structure factor for light scattering”* (34).

Second half of Part I of this thesis will deal with this one dimensional theory, being the corner stone of present work on ensembles, experimental as well as theoretical. Then the problems concerning this simplified theory will be outlined. As we shall discover, diffraction is poorly dealt with in the typical QM description. In Part II we derive the “proper quantum model”, where our onset will be classical field theory. We will find that in this line, the final quantum theory will compare to the macroscopic Maxwell equations. In other words, we derive a theory describing QM in a media. In Part III we consider the scattering induced dynamics within an ensemble. We will find that nontrivial and non-accounted-for processes occur.

One-Dimensional Theory

In this chapter we will derive an effective Hamiltonian describing the interaction between a free space laserfield and an atomic gas. This theory at onset is not necessarily one-dimensional, but considering the incident light as single mode and neglecting three-dimensional processes such as, one atom absorbing a photon, then re-emitting a photon that gets absorbed by another atom, effectively leaves one with a one-dimensional theory, a theory describing a one dimensional system. This theory is an essential result on ensemble theory in the frames of quantum optics, and therefore an indispensable part in any thesis on the subject. It is however its fundamental incompleteness that is the motivation for making the three-dimensional theory, taking all such scattering processes into account.

We consider the atomic gas as consisting of N identical atoms, where we treat the atoms in the *one electron approximation*. This means that the atom consist of a heavy nuclei of effective charge $e = +1$ with a much lighter valence electron of effective charge $e = -1$. The electric field will be treated in the Coulomb Gauge, and we further more use the *dipole approximation*. This means that we assume the spatial extent of the atom to be much smaller than the wavelength of the light. In these frames the minimal coupling Hamiltonian reduce* and the interaction Hamiltonian between free-space light and atoms, may be written as

$$\mathcal{H}_{\text{int}} = - \int d^3r \hat{\mathbf{E}}(\mathbf{r}, t) \cdot \hat{\mathbf{P}}(\mathbf{r}, t), \quad (2.1.1)$$

$\hat{\mathbf{P}}(\mathbf{r}, t)$ being the dipole operator for the atom located at position \mathbf{r} , and $\hat{\mathbf{E}}(\mathbf{r}, t)$ being the electric

*This derivation is found in many textbooks see, e.g., (4)

field at the given position.

2.2 Effective Hamiltonian

The basic assumption on the atoms is that they will submit to the level scheme as shown in Figure 2.1 We consider a system of two groundlevels and two excited levels, each of angular momentum $\frac{1}{2}$ and magnetic moment $\pm\frac{1}{2}$. The excited energy levels are equal in energy as is the groundlevels. Our

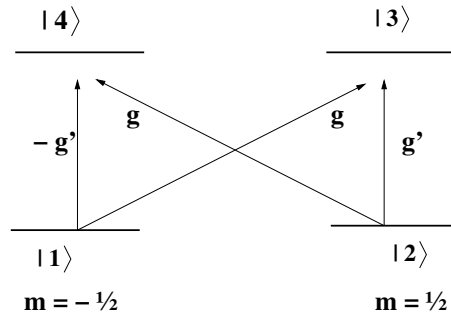


Figure 2.1: The atomic system, labeling of the states

desire is to reduce the interaction Hamiltonian (2.1.1) to an effective one, considering only energy conserving terms. This generally means that we wish to work in the *Rotating Wave Approximation*(RWA).[†] In equation (2.2.1) we define the basis in which we will perform this reduction.

$$\mathbf{e}_+ = \frac{\mathbf{e}_x + i\mathbf{e}_y}{\sqrt{2}}, \quad \mathbf{e}_- = \frac{\mathbf{e}_x - i\mathbf{e}_y}{\sqrt{2}}, \quad \mathbf{e}_0 = \mathbf{e}_z, \quad (2.2.1)$$

$\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ being the usual Euclidean basis vectors. An arbitrary vector may then be written in this basis as a linear combination of spherical tensors of rank one.[‡]

$$\mathbf{r} = V_{-1}^{(1)} \mathbf{e}_+ - V_{+1}^{(1)} \mathbf{e}_- + V_0^{(1)} \mathbf{e}_0 \quad (2.2.2)$$

In Appendix B.1.1 is written the scalar products and cross products between the basisvectors.

The Dipole Operator $\hat{\mathbf{P}}$

Let us look at the dipole operator. This is for the j -th atom given as the vector spacing between the valence electron and the nucleus times the electron charge e . Expanding the dipole operator in the

[†]See, e.g., (4) for discussion of this approximation

[‡]See, e.g., (35)

basis given by the intrinsic angular momentum eigenfunctions for the atom gives the following.

$$\hat{\mathbf{P}}(\mathbf{r}, t) = \sum_{nm \in \{1,2,3,4\}} \sum_j^{\text{Atoms}} \delta(\mathbf{r} - \mathbf{r}_j) |n\rangle_j \langle n|_j \mathbf{d}_j |m\rangle_j \langle m|_j \quad (2.2.3)$$

The vector \mathbf{d}_j is the before mentioned vector spacing times charge. By writing this vector as a linear combination of spherical tensors of rank one as done in equation (2.2.2), we may calculate the matrix elements $\langle n|_j \mathbf{d}_j |m\rangle_j$, using the Wigner-Eckart theorem as well as Clebsch-Gordan coefficients. In this respect the dipole operator $\hat{\mathbf{P}}(\mathbf{r}, t)$ may be written as:

$$\hat{\mathbf{P}}(\mathbf{r}, t) = \sum_j^{\text{Atoms}} \delta^3(\mathbf{r} - \mathbf{r}_j) g d [(\hat{\sigma}_{13}^j + \hat{\sigma}_{42}^j) \mathbf{e}_+ + (\hat{\sigma}_{31}^j - \hat{\sigma}_{24}^j) \mathbf{e}_- + \frac{1}{\sqrt{2}} (\hat{\sigma}_{32}^j + \hat{\sigma}_{23}^j - \hat{\sigma}_{14}^j - \hat{\sigma}_{41}^j) \mathbf{e}_0]. \quad (2.2.4)$$

We used the shorthand notation $\hat{\sigma}_{nm}^j \equiv |n\rangle_j \langle m|_j$, and g is a coupling constant depending only on properties of ground- and excited levels, d is the characteristic dipole moment of the system.[§] A detailed calculation of the various matrix elements have been put in Appendix B.2.1.

Electric Field

The electric field, we will expand in transverse plane waves.[¶]

$$\hat{\mathbf{E}}(\mathbf{r}, t) = \hat{\mathbf{E}}^-(\mathbf{r}, t) + \hat{\mathbf{E}}^+(\mathbf{r}, t) \quad (2.2.5)$$

where

$$\hat{\mathbf{E}}^-(\mathbf{r}, t) = i \sum_{\mathbf{k}s} \left(\frac{\hbar \omega_k}{2 \epsilon_0 V} \right)^{\frac{1}{2}} \hat{a}_{\mathbf{k}s}(0) \mathbf{e}_{\mathbf{k}s} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} \quad (2.2.6)$$

$$\hat{\mathbf{E}}^+(\mathbf{r}, t) = -i \sum_{\mathbf{k}s} \left(\frac{\hbar \omega_k}{2 \epsilon_0 V} \right)^{\frac{1}{2}} \hat{a}_{\mathbf{k}s}^\dagger(0) \mathbf{e}_{\mathbf{k}s}^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)}. \quad (2.2.7)$$

V is the volume of the quantization box. \mathbf{k} is the vector describing wavelength and propagation direction of the particular field component. $\mathbf{e}_{\mathbf{k}s}$ is the basis vectors expanding the plane orthogonal to the vector \mathbf{k} , where $s \in \{1, 2\}$. i is the complex i , and the relation between ω_k and \mathbf{k} is the simple one, $|\mathbf{k}| = \frac{\omega_k}{c}$, c being the speed of light in vacuum. Writing the electric field in the rotating basis,

[§]These could of course have been written as one coupling constant.

[¶]See, e.g, (36) for derivation.

gives the following:

$$\hat{\mathbf{E}}^+(\mathbf{r}, t) = \hat{E}_-^+ \mathbf{e}_+ + \hat{E}_+^+ \mathbf{e}_- + \hat{E}_0^+ \mathbf{e}_0, \quad (2.2.8a)$$

$$\hat{\mathbf{E}}^-(\mathbf{r}, t) = \hat{E}_-^- \mathbf{e}_+ + \hat{E}_+^- \mathbf{e}_- + \hat{E}_0^- \mathbf{e}_0 \quad (2.2.8b)$$

To shorten the notation we did not write explicitly the space and time dependence on the coefficients this is implied. It is simple to check that in fact the electric field in this basis is still a hermitian operator, hence $(\hat{\mathbf{E}}^+(\mathbf{r}, t))^\dagger = \hat{\mathbf{E}}^-(\mathbf{r}, t)$. In the simple case of a single mode field of wave vector \mathbf{k}_L we see that we may extract the time evolution of the field components entirely. Let us therefore assume that the electric field is in fact a realistic laser field, driven primarily with some frequency ω_L . Extracting a term oscillating with this frequency in time reduce the remaining part to a slow oscillating one. Slowly oscillating operators will be denoted by tilde.

$$\hat{\mathbf{E}}(\mathbf{r}, t) = \tilde{\mathbf{E}}^-(\mathbf{r}, t)e^{i\omega_L t} + \tilde{\mathbf{E}}^+(\mathbf{r}, t)e^{-i\omega_L t}. \quad (2.2.9)$$

We are now ready to return to the Hamiltonian.

The Hamiltonian describing the free field is given by^{||}

$$\hat{\mathcal{H}}_{\text{Field}} = \sum_{\mathbf{k}_s} \hbar \omega_k \hat{a}_{\mathbf{k}_s}^\dagger(t) \hat{a}_{\mathbf{k}_s}(t) \quad (2.2.10)$$

and the Hamiltonian describing the free atoms is

$$\begin{aligned} \hat{\mathcal{H}}_{\text{Atoms}} &= \sum_j^{\text{Atoms}} \sum_{n \in \{1,2,3,4\}} E_n^j \hat{\sigma}_{nn}^j \\ &= \sum_j^{\text{Atoms}} \frac{1}{2} \hbar \omega_0 (\hat{\sigma}_{33}^j - \hat{\sigma}_{11}^j + \hat{\sigma}_{44}^j - \hat{\sigma}_{22}^j) + (\text{zero point energy terms}) \end{aligned} \quad (2.2.11)$$

The Bohr frequency is defined as $\omega_0 = \frac{E_{4,3}^j - E_{2,1}^j}{\hbar}$ and $0 = \frac{E_4^j - E_3^j}{\hbar} = \frac{E_2^j - E_1^j}{\hbar}$. The zero point energy terms we will neglect, hence we those our potential zero at the ground level, and thus our full

^{||}This can be found in almost any standard textbook on quantum optics.

Hamiltonian is given by:

$$\begin{aligned}
\hat{\mathcal{H}} &= \hat{\mathcal{H}}_{\text{Field}} + \hat{\mathcal{H}}_{\text{Atoms}} + \hat{\mathcal{H}}_{\text{Int}} \\
&= \sum_{\mathbf{k}s} \hbar\omega_k \hat{a}_{\mathbf{k}s}^\dagger(t) \hat{a}_{\mathbf{k}s}(t) + \sum_j^{\text{Atoms}} \frac{1}{2} \hbar\omega_0 (\hat{\sigma}_{33}^j - \hat{\sigma}_{11}^j + \hat{\sigma}_{44}^j - \hat{\sigma}_{22}^j) \\
&\quad - \sum_j^{\text{Atoms}} g d \left((\hat{E}_-^+ + \hat{E}_-^-) (\hat{\sigma}_{31}^j + \hat{\sigma}_{24}^j) + (\hat{E}_+^+ + \hat{E}_+^-) (\hat{\sigma}_{13}^j + \hat{\sigma}_{42}^j) \right) \\
&\quad + \frac{1}{\sqrt{2}} (\hat{E}_0^+ + \hat{E}_0^-) (\hat{\sigma}_{32}^j + \hat{\sigma}_{23}^j - \hat{\sigma}_{14}^j - \hat{\sigma}_{41}^j)
\end{aligned} \tag{2.2.12}$$

In the RWA we neglect rapidly oscillating terms in the Hamiltonian, this means we will neglect terms primarily oscillating with a frequency different from $\pm\Delta$, where we define $\Delta \equiv \omega_0 - \omega_L$. The primary time evolution of the atomic operators $\hat{\sigma}_{nm}^j$ is governed by $\hat{\mathcal{H}}_{\text{Atoms}}$. This follows by using The Heisenberg equation of motion and neglecting the interaction Hamiltonian. *Heisenberg Equation of Motion*

$$\frac{d}{dt} \hat{A}(t) = \frac{i}{\hbar} [\hat{\mathcal{H}}; \hat{A}(t)] \tag{2.2.13}$$

This shows us the following:

$$\begin{aligned}
\frac{d}{dt} \hat{\sigma}_{nm}^j &\approx -i\omega_0 \hat{\sigma}_{nm}^j \quad \text{where } n \in \{1, 2\}, m \in \{3, 4\} \\
\frac{d}{dt} \hat{\sigma}_{nm}^j &\approx i\omega_0 \hat{\sigma}_{nm}^j \quad \text{where } n \in \{3, 4\}, m \in \{1, 2\}
\end{aligned} \tag{2.2.14}$$

Having established information enough to actually perform the RWA we arrive at the following Hamiltonian.

$$\begin{aligned}
\hat{\mathcal{H}} &= \sum_{\mathbf{k}s} \hbar\omega_k \hat{a}_{\mathbf{k}s}^\dagger(t) \hat{a}_{\mathbf{k}s}(t) + \sum_j^{\text{Atoms}} \frac{1}{2} \hbar\omega_0 (\hat{\sigma}_{33}^j - \hat{\sigma}_{11}^j + \hat{\sigma}_{44}^j - \hat{\sigma}_{22}^j) \\
&\quad - \sum_j^{\text{Atoms}} g d \left(\hat{E}_-^+ \hat{\sigma}_{24}^j + \hat{E}_-^- \hat{\sigma}_{31}^j + \hat{E}_+^+ \hat{\sigma}_{13}^j + \hat{E}_+^- \hat{\sigma}_{42}^j + \frac{1}{\sqrt{2}} (\hat{E}_0^+ \hat{\sigma}_{23}^j + \hat{E}_0^- \hat{\sigma}_{32}^j - \hat{E}_0^+ \hat{\sigma}_{14}^j - \hat{E}_0^- \hat{\sigma}_{41}^j) \right)
\end{aligned} \tag{2.2.15}$$

Again for simplicity we did not explicitly write the time and space dependence on the field components. These shall be evaluated at time t and space \mathbf{r}_j . In the following we will make an adiabatic elimination of the excited levels of the atoms. This means that we expect the lifetime of an excited atom to be so short that for a far detuned laser the population of atoms in excited states are negligible. The average time it takes a far detuned laser to excite an atom is much longer than the lifetime of an excited atom. This elimination basically means that we detect changes in the population of

groundlevels, and the fact that the atom, undergoing this change, went through an excited state, we ignore.**

2.2.1 Adiabatic Elimination

In this section we let the operators describing coherency between ground and excited states $\hat{\sigma}_{nm}$ follow the ground states and light field adiabatically. For the operator $\hat{\sigma}_{24}$ we find using Heisenberg equation (2.2.13) and at same stroke neglecting processes happening at excited levels

$$\frac{d}{dt}\hat{\sigma}_{24}^j = \frac{i}{\hbar}[\hat{\mathcal{H}}; \hat{\sigma}_{24}^j] = -i\frac{gd}{\hbar}(-\hat{E}_+^-\hat{\sigma}_{22}^j + \hat{E}_0^-\hat{\sigma}_{21}^j) - i\omega_0\hat{\sigma}_{24}^j \quad (2.2.16)$$

The general solution to this differential equation is given by

$$\hat{\sigma}_{24}^j = e^{-i\omega_0 t} \left(1 + \int_{t_0}^t dt' \left[-i\frac{gd}{\hbar} e^{i\omega_0 t'} (-\hat{E}_+^-\hat{\sigma}_{22}^j + \frac{1}{\sqrt{2}}\hat{E}_0^-\hat{\sigma}_{21}^j) \right] \right). \quad (2.2.17)$$

Keeping only slowly oscillating terms, and using the primary time evolution of the fields given in equation (2.2.9), we get

$$\hat{\sigma}_{24}^j = \frac{gd}{\hbar\Delta} \left(\hat{E}_+^-\hat{\sigma}_{22}^j - \frac{1}{\sqrt{2}}\hat{E}_0^-\hat{\sigma}_{21}^j \right). \quad (2.2.18)$$

Atomic Spin operator $\hat{\mathbf{J}}^j$

It is a good idea to define the spin operators:

$$\hat{J}_+^j = \frac{1}{\sqrt{2}}\hat{\sigma}_{21}^j, \quad \hat{J}_-^j = \frac{1}{\sqrt{2}}\hat{\sigma}_{12}^j, \quad \hat{J}_0^j = \frac{1}{2}(\hat{\sigma}_{22}^j - \hat{\sigma}_{11}^j) \quad (2.2.19)$$

Where the spin operators in the rotating basis are given by

$$\hat{j}_+^j = \frac{\hat{J}_x^j + i\hat{J}_y^j}{\sqrt{2}}, \quad \hat{j}_-^j = \frac{\hat{J}_x^j - i\hat{J}_y^j}{\sqrt{2}}, \quad \hat{j}_0^j = \hat{J}_z^j. \quad (2.2.20)$$

This means that the vector operator $\hat{\mathbf{J}}^j$ is written in the rotating basis as

$$\hat{\mathbf{J}}^j = \hat{j}_-^j \mathbf{e}_+ + \hat{j}_+^j \mathbf{e}_- + \hat{j}_0^j \mathbf{e}_0. \quad (2.2.21)$$

**The use of this technique is described, e.g., in (37).

Using these definitions we may rewrite equation (2.2.18) and get

$$\hat{\sigma}_{24}^j = \frac{gd}{\hbar\Delta} (\hat{E}_+^- \hat{\sigma}_{22}^j - \hat{E}_0^- \hat{J}_+^j). \quad (2.2.22)$$

In Appendix B.3 equation (B.3.1) the remaining atomic coherence operators are written. Inserting these into equation (2.2.15) we find the effective interaction Hamiltonian to be

$$\begin{aligned} \hat{\mathcal{H}}_{\text{Eff}} = & - \sum_j^{\text{Atoms}} \frac{g^2 d^2}{\hbar\Delta} \left\{ \hat{E}_+^+ (\hat{E}_+^- \hat{\sigma}_{22}^j - \hat{E}_0^- \hat{J}_+^j) + \hat{E}_-^- (\hat{E}_+^+ \hat{\sigma}_{11}^j + \hat{E}_0^+ \hat{J}_+^j) + \hat{E}_+^+ (\hat{E}_-^- \hat{\sigma}_{11}^j + \hat{E}_0^- \hat{J}_-^j) \right. \\ & + \hat{E}_+^+ (\hat{E}_-^- \hat{\sigma}_{22}^j - \hat{E}_0^+ \hat{J}_-^j) + \frac{1}{\sqrt{2}} \left[\hat{E}_0^+ \left(\frac{1}{\sqrt{2}} \hat{E}_-^- \hat{\sigma}_{22}^j + \sqrt{2} \hat{E}_-^- \hat{J}_+^j \right) + \hat{E}_0^- \left(\frac{1}{\sqrt{2}} \hat{E}_+^+ \hat{\sigma}_{22}^j + \sqrt{2} \hat{E}_+^+ \hat{J}_-^j \right) \right. \\ & \left. \left. - \hat{E}_0^+ \left(\frac{1}{\sqrt{2}} \hat{E}_-^- \hat{\sigma}_{11}^j - \sqrt{2} \hat{E}_-^- \hat{J}_-^j \right) - \hat{E}_0^- \left(\frac{1}{\sqrt{2}} \hat{E}_+^+ \hat{\sigma}_{11}^j - \sqrt{2} \hat{E}_+^+ \hat{J}_+^j \right) \right] \right\} \quad (2.2.23) \end{aligned}$$

This equation can be reduced to a short and for the eye very appealing result. The detailed calculation can be found in Appendix B.3.1

$$\hat{\mathcal{H}}_{\text{Eff}} = - \sum_j^{\text{Atoms}} \frac{2g^2 d^2}{\hbar\Delta} \hat{\mathbf{E}}^+(\mathbf{r}_j, t) \bar{\alpha}^j \hat{\mathbf{E}}^-(\mathbf{r}_j, t) \quad (2.2.24)$$

where

$$\bar{\alpha}^j \equiv \frac{1}{2} \bar{\mathbf{I}}^j - i \hat{\mathbf{J}}^j \times . \quad (2.2.25)$$

$\bar{\mathbf{I}}^j$ is the atomic identity given by $\bar{\mathbf{I}}^j = \hat{\sigma}_{22} + \hat{\sigma}_{11}$.

Considering the fact that the Hamiltonian is a tensor of zero rank, and the system in consideration involves only the fields $\hat{\mathbf{J}}$, $\hat{\mathbf{E}}^+$ and $\hat{\mathbf{E}}^-$, the possible combinations of vector operations, leading to a tensor of zero rank is exactly the kind found in equation (2.2.24). From the hermicity of the Hamiltonian we find a factor of complex i on the term: $ig \hat{\mathbf{E}}^+ \cdot (\hat{\mathbf{J}} \times \hat{\mathbf{E}}^-)$. When one also takes into consideration the conservation of momentum, the Hamiltonian (2.2.24) is not surprising. Our contribution was thus the factor of 2 and the sign.^{††}

^{††}Ironically as it may seem, this is often an extremely difficult task.

2.3 Equations of Motion

In this section we will describe the light matter interaction to lowest order. It will be clear what we mean by lowest order, and the derivation are also meant as an illustration of the problems the full theory will deal with.

The dispersive effect of the ensemble is not negligible, however in the simple case with a homogeneous ensemble this effect can easily be implemented. We will make the Hamiltonian seemingly continuous by introducing the following scalar and vector fields. Our goal being to treat the system continuously.

$$\rho = \langle \sum_j^{\text{Atoms}} \delta(\mathbf{r} - \mathbf{r}_j) \rangle_{\text{Mean}} \quad \text{and} \quad \lambda(\mathbf{r}) = \sum_j^{\text{Atoms}} \delta(\mathbf{r} - \mathbf{r}_j) - \rho \quad (2.3.1)$$

$$\hat{\mathbf{J}}(\mathbf{r}, t) = \sum_j^{\text{Atoms}} \delta(\mathbf{r} - \mathbf{r}_j) \hat{\mathbf{J}}^j. \quad (2.3.2)$$

Using these definitions in our Hamiltonian leaves us with,

$$\begin{aligned} \hat{\mathcal{H}}_{\text{Eff}} = & - \int d^3 r \frac{2g^2 d^2}{\hbar \Delta} \hat{\mathbf{E}}^+(\mathbf{r}, t) \left(\frac{1}{2} \lambda(\mathbf{r}) - i \hat{\mathbf{J}}(\mathbf{r}, t) \times \right) \hat{\mathbf{E}}^-(\mathbf{r}, t) \\ & - \int d^3 r \frac{g^2 d^2 \rho}{\hbar \Delta} \hat{\mathbf{E}}^+(\mathbf{r}, t) \hat{\mathbf{E}}^-(\mathbf{r}, t) \end{aligned} \quad (2.3.3)$$

The last term we may reduce in the following manner, using equations (2.2.5)

$$\begin{aligned} & - \int d^3 r \frac{g^2 d^2 \rho}{\hbar \Delta} \hat{\mathbf{E}}^+(\mathbf{r}, t) \hat{\mathbf{E}}^-(\mathbf{r}, t) \\ & = - \int d^3 r \frac{g^2 d^2 \rho}{\hbar \Delta} \sum_{\substack{\mathbf{k}_s \\ \mathbf{k}'_{s'}}} \sqrt{\frac{\hbar \omega_k}{2\epsilon_0 V}} \sqrt{\frac{\hbar \omega_{k'}}{2\epsilon_0 V}} \mathbf{e}_{\mathbf{k}_s} \cdot \mathbf{e}_{\mathbf{k}'_{s'}}^* \hat{a}_{\mathbf{k}_s}^\dagger \hat{a}_{\mathbf{k}'_{s'}} e^{i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{r}} e^{i(\omega_k - \omega_{k'})t} \\ & = - \sum_{\substack{\mathbf{k}_s \\ \mathbf{k}'_{s'}}} \frac{g^2 d^2 \rho \hbar}{2\hbar \Delta \epsilon_0} \sqrt{\omega_k \omega_{k'}} \mathbf{e}_{\mathbf{k}_s} \cdot \mathbf{e}_{\mathbf{k}'_{s'}}^* \hat{a}_{\mathbf{k}_s}^\dagger \hat{a}_{\mathbf{k}'_{s'}} \delta_{\mathbf{k}\mathbf{k}'} e^{i(\omega_k - \omega_{k'})t} \\ & = - \sum_{\mathbf{k}_s} \sum_{s'} \hbar \frac{g^2 d^2 \rho}{2\hbar \Delta \epsilon_0} \omega_k (\mathbf{e}_{\mathbf{k}_s} \cdot \mathbf{e}_{\mathbf{k}'_{s'}}^*) \hat{a}_{\mathbf{k}_s}^\dagger \hat{a}_{\mathbf{k}'_{s'}} \end{aligned}$$

using the orthonormality of the polarization vectors, and setting $\chi = -\frac{g^2 d^2 \rho}{2\hbar \Delta \epsilon_0}$ we find

$$\begin{aligned} &= \sum_{\mathbf{k}_s} \sum_{s'} \hbar \chi \omega_k \delta_{ss'} \hat{a}_{\mathbf{k}_s}^\dagger \hat{a}_{\mathbf{k}_s'} \\ &= \sum_{\mathbf{k}_s} \hbar \chi \omega_k \hat{a}_{\mathbf{k}_s}^\dagger \hat{a}_{\mathbf{k}_s} \end{aligned} \quad (2.3.4)$$

This term we include in the Hamiltonian describing the field, (2.2.10) and get the modified free field Hamilton.

$$\mathcal{H}_{\text{Field}} = \sum_{\mathbf{k}_s} \hbar (1 + \chi) \omega_k \hat{a}_{\mathbf{k}_s}^\dagger \hat{a}_{\mathbf{k}_s} \equiv \sum_{\mathbf{k}_s} \hbar (\sqrt{\epsilon})^{-1} \omega_k \hat{a}_{\mathbf{k}_s}^\dagger \hat{a}_{\mathbf{k}_s} \quad (2.3.5)$$

The effect merely introduces a difference in propagation speed of light between the ensemble and free space. This is calculated in Appendix B.4.4.

2.3.1 Commutation relation on the Spin operator

The following section is devoted to describing the commutation relations on the spin operator. As previously mentioned we will treat the system continuous, meaning the spin of the system. This introduces the necessary continuous spin commutation relations.

We defined, in equation (2.2.19) and (2.2.20), the atomic spin operator. The commutation relation to which they apply is the standard one, showed in Appendix B.4.2.

$$[\hat{J}_n^j, \hat{J}_m^{j'}] = i \epsilon_{nml} \hat{J}_l^j \delta_{jj'} \quad (2.3.6)$$

Commutation relations, regarding the field formulation of the atomic spin, can then be found to give the following:

$$[\hat{J}_m(\mathbf{r}); \hat{J}_n(\mathbf{r}')] = i \delta(\mathbf{r} - \mathbf{r}') \epsilon_{nml} \hat{J}_l(\mathbf{r}). \quad (2.3.7)$$

Detailed calculations have been put in Appendix B.4.2. In the process of reducing the system to one dimension, we will consider the spin of the atoms in the x -, and y -plane to be uncorrelated and we define the operator: *normalized "slice" of spin*(25).

$$\hat{J}_n(z, t) \equiv \lim_{\delta z \rightarrow 0} \int_z^{z+\delta z} dz' \iint dx' dy' \frac{1}{\rho A \delta z} \hat{J}_n(\mathbf{r}', t) \quad (2.3.8)$$

These spin operators will apply to a similar set of commutation relation. Again the detailed calcu-

lations can be found in Appendix B.4.2.

$$[\hat{J}_n(z, t); \hat{J}_m(z', t)] = \frac{i}{\rho A} \delta(z - z') \varepsilon_{nml} \hat{J}_l(z, t) \quad (2.3.9)$$

Now we continue working with the Hamiltonian as we left it in equation (2.3.3). We assume that the field $\hat{\mathbf{E}}(\mathbf{r}, t)$ does not change over a slice of the ensemble at any given position z , and we thus write.

$$\begin{aligned} \hat{\mathcal{H}}_{\text{Eff}}^{\text{Int}} &= - \int_0^L dz \frac{2g^2 d^2}{\hbar \Delta} \hat{\mathbf{E}}^+(\mathbf{r}, t) \cdot \left(\frac{1}{2} \left\{ \iint dxdy \lambda(\mathbf{r}) \right\} - i \left\{ \iint dxdy \hat{\mathbf{J}}(\mathbf{r}, t) \right\} \times \right) \hat{\mathbf{E}}^-(\mathbf{r}, t) \\ &= - \int_0^L dz \frac{2g^2 d^2 \rho A}{\hbar \Delta} \hat{\mathbf{E}}^+(\mathbf{r}, t) \cdot \left(\frac{1}{2} \left\{ \lim_{\delta z \rightarrow 0} \int_z^{z+\delta z} dz \iint dxdy \lambda(\mathbf{r}) \right\} \right. \\ &\quad \left. - i \left\{ \lim_{\delta z \rightarrow 0} \int_z^{z+\delta z} dz \iint dxdy \hat{\mathbf{J}}(\mathbf{r}, t) \right\} \times \right) \hat{\mathbf{E}}^-(\mathbf{r}, t) \end{aligned} \quad (2.3.10)$$

$$\begin{aligned} &= - \int_0^L dz \frac{2g^2 d^2 \rho A}{\hbar \Delta} \hat{\mathbf{E}}^+(\mathbf{r}, t) \cdot \underbrace{\left(\frac{1}{2} \left\{ \lim_{\delta z \rightarrow 0} \int_z^{z+\delta z} dz \iint dxdy \lambda(\mathbf{r}) \right\} - i \hat{\mathbf{J}}(z, t) \times \right)}_* \hat{\mathbf{E}}^-(\mathbf{r}, t) \\ &= \int_0^L dz \frac{i2g^2 d^2 \rho A}{\hbar \Delta} \hat{\mathbf{E}}^+(\mathbf{r}, t) \cdot (\hat{\mathbf{J}}(z, t) \times \hat{\mathbf{E}}^-(\mathbf{r}, t)) \end{aligned} \quad (2.3.11)$$

The term (*) cancels when making the integration, since we subtracted the very effect of dispersion. Our continuous Hamiltonian, capable of describing a homogeneous ensemble, thus reads:

$$\hat{\mathcal{H}}_{\text{Eff}}^{\text{Int}} = \int_0^L \frac{i2g^2 d^2}{\hbar \Delta} \hat{\mathbf{E}}^+(\mathbf{r}, t) \cdot (\hat{\mathbf{J}}(z, t) \times \hat{\mathbf{E}}^-(\mathbf{r}, t)) \rho A dz. \quad (2.3.12)$$

1-dimensional Electric field

The electric field is generally given as an infinite sum over all excited modes the field might enter.

$$\mathbf{E}(\mathbf{r}, t) = i \sum_{\mathbf{k}_s} \sqrt{\frac{\hbar \omega_k}{2\epsilon_0 V}} (\mathbf{e}_{\mathbf{k}_s} \hat{a}_{\mathbf{k}_s} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} - \mathbf{e}_{\mathbf{k}_s}^* \hat{a}_{\mathbf{k}_s}^\dagger e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)}) \quad (2.3.13)$$

However, under certain circumstances we may approximate the light field with a one-dimensional theory describing a propagating light field. (25; 38) This is the case with the dispersive object being of Fresnel number unity. Assuming the field to propagate along the z -axis, and the carrier frequency to be ω_L , we write the electric field in the continuous version as:

$$\mathbf{E}(\mathbf{r}, t) = \sum_s \sqrt{\frac{\hbar\omega_L}{2\epsilon_0 A}} (\hat{a}_s(z, t)\mathbf{e}_s + \hat{a}_s^\dagger(z, t)\mathbf{e}_s^*). \quad (2.3.14)$$

The detailed calculation can be found in appendix B.4.1, also see article (39). The commutation relations applying to the continuous creation and annihilation operators are:

$$[\hat{a}_s(z, t); \hat{a}_{s'}^\dagger(z', t)] = \delta_{ss'} \delta(z - z'). \quad (2.3.15)$$

If we insert into our Hamiltonian (2.3.12) the continuous one dimensional laser field, (2.3.14) our Hamiltonian reduce to the following. We will also choose our polarization, such that \mathbf{e}_1 is in the x -direction, and \mathbf{e}_2 in the y -direction.

$$\begin{aligned} \mathcal{H}_{\text{Eff}}^{\text{Int}} &= \int_0^L \frac{i2g^2 d^2}{\hbar\Delta} \hat{\mathbf{E}}^+(\mathbf{r}, t) \cdot (\hat{\mathbf{J}}(z, t) \times \hat{\mathbf{E}}^-(\mathbf{r}, t)) \rho A dz \\ &= \int_0^L \frac{ig^2 d^2 \omega_L}{\Delta\epsilon_0 A} (\hat{a}_1^\dagger(z, t)\mathbf{e}_1 + \hat{a}_2^\dagger(z, t)\mathbf{e}_2) \cdot \{\hat{\mathbf{J}}(z, t) \times (\hat{a}_1(z, t)\mathbf{e}_1 + \hat{a}_2(z, t)\mathbf{e}_2)\} \rho A dz \\ &= \int_0^L \frac{ig^2 d^2 \omega_L \rho}{\Delta\epsilon_0} \{-\hat{a}_1^\dagger(z, t)\hat{a}_2(z, t)\hat{J}_z(z, t) + \hat{a}_2^\dagger(z, t)\hat{a}_1(z, t)\hat{J}_z(z, t)\} dz \end{aligned} \quad (2.3.16)$$

Let us introduce local Stokes operators, defined by:

$$\hat{S}_x(z, t) = \frac{1}{2}(a_1^\dagger(z, t)\hat{a}_1(z, t) - a_2^\dagger(z, t)\hat{a}_2(z, t)), \quad (2.3.17a)$$

$$\hat{S}_y(z, t) = \frac{1}{2}(a_1^\dagger(z, t)\hat{a}_2(z, t) + a_2^\dagger(z, t)\hat{a}_1(z, t)), \quad (2.3.17b)$$

$$\hat{S}_z(z, t) = \frac{1}{2i}(a_1^\dagger(z, t)\hat{a}_2(z, t) - a_2^\dagger(z, t)\hat{a}_1(z, t)). \quad (2.3.17c)$$

The local stokes operators apply to the following commutations relations, as is shown in Appendix B.4.3.

$$[\hat{S}_n(z, t); \hat{S}_m(z', t)] = i\delta(z - z') \epsilon_{nml} \hat{S}_m(z, t) \quad (2.3.18)$$

Inserting the local Stoke operators into equation (2.3.16), one finally arrives at,

$$\mathcal{H}_{\text{Eff}}^{\text{Int}} = \int_0^L \frac{g^2 d^2 \omega_L \rho}{\Delta\epsilon_0} \hat{S}_z(z, t) \hat{J}_z(z, t) dz. \quad (2.3.19)$$

2.3.2 Equations of motion

In the following we will derive equations of motion first regarding the spin of the atom, and secondly for the Stokes operators. The Heisenberg equation of motion gives, for the spin of the atom, the following equations of motion.

$$\begin{aligned}
\frac{\partial}{\partial t} \hat{J}_x(z, t) &= \frac{i}{\hbar} [\mathcal{H}_{\text{EF}}^{\text{Int}}; \hat{J}_x(z, t)] \\
&= \frac{i}{\hbar} \int_0^L dz' \frac{g^2 d^2 \omega_L \rho}{\Delta \epsilon_0} \hat{S}_z(z', t) [\hat{J}_z(z', t); \hat{J}_x(z, t)] \\
&= i \frac{g^2 d^2 \omega_L \rho}{\hbar \Delta \epsilon_0} \int_0^L dz' \hat{S}_z(z', t) \frac{i}{\rho A} \delta(z - z') \hat{J}_y(z, t) \\
&= - \frac{g^2 d^2 \omega_L}{\hbar \Delta \epsilon_0 A} \hat{S}_z(z', t) \hat{J}_y(z, t)
\end{aligned} \tag{2.3.20a}$$

where we used the commutation relations (2.3.9) Similarly we find:

$$\begin{aligned}
\frac{\partial}{\partial t} \hat{J}_y(z, t) &= \frac{i}{\hbar} [\mathcal{H}_{\text{EF}}^{\text{Int}}; \hat{J}_y(z, t)] \\
&= \frac{g^2 d^2 \omega_L}{\hbar \Delta \epsilon_0 A} \hat{S}_z(z', t) \hat{J}_x(z, t)
\end{aligned} \tag{2.3.20b}$$

and,

$$\frac{\partial}{\partial t} \hat{J}_z(z, t) = \frac{i}{\hbar} [\mathcal{H}_{\text{EF}}^{\text{Int}}; \hat{J}_z(z, t)] = 0 \tag{2.3.20c}$$

We also find equations of motion for the Stokes operators. The derivation of this particular form of equation of motion is left in Appendix B.4.4

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + \frac{c}{\sqrt{\epsilon}} \frac{\partial}{\partial z} \right) \hat{S}_x(z, t) &= \frac{i}{\hbar} [\mathcal{H}_{\text{EF}}^{\text{Int}}; \hat{S}_x(z, t)] \\
&= \frac{i}{\hbar} \int_0^L dz' \frac{g^2 d^2 \omega_L \rho}{\Delta \epsilon_0} \hat{J}_z(z', t) [\hat{S}_z(z', t); \hat{S}_x(z, t)] \\
&= \frac{i g^2 d^2 \omega_L \rho}{\hbar \Delta \epsilon_0} \int_0^L dz' \hat{J}_z(z', t) i \delta(z - z') \hat{S}_y(z, t) \\
&= - \frac{g^2 d^2 \omega_L \rho}{\hbar \Delta \epsilon_0} \hat{J}_z(z, t) \hat{S}_y(z, t)
\end{aligned} \tag{2.3.21a}$$

again we used the commutation relation (2.3.18). Same procedure gives,

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \frac{c}{\sqrt{\epsilon}} \frac{\partial}{\partial z}\right) \hat{S}_y(z, t) &= \frac{i}{\hbar} [\mathcal{H}_{\text{Eff}}^{\text{Int}}; \hat{S}_y(z, t)] \\ &= \frac{g^2 d^2 \omega_L \rho}{\hbar \Delta \epsilon_0} \hat{J}_z(z, t) \hat{S}_x(z, t) \end{aligned} \quad (2.3.21b)$$

and

$$\left(\frac{\partial}{\partial t} + \frac{c}{\sqrt{\epsilon}} \frac{\partial}{\partial z}\right) \hat{S}_y(z, t) = \frac{i}{\hbar} [\mathcal{H}_{\text{Eff}}^{\text{Int}}; \hat{S}_y(z, t)] = 0 \quad (2.3.21c)$$

To simplify the equations we change to the variable $\tau = t - \frac{z\sqrt{\epsilon}}{c}$, and define the constant $\kappa \equiv \frac{g^2 d^2 \omega_L \sqrt{\epsilon}}{\hbar \Delta \epsilon_0 A}$. This gives us the following equations.

$$\frac{\partial}{\partial \tau} \hat{J}_x(z, \tau) = -\kappa \hat{J}_y(z, \tau) \frac{c}{\sqrt{\epsilon}} \hat{S}_z(z, \tau), \quad \frac{\partial}{\partial \tau} \hat{J}_y(z, \tau) = \kappa \hat{J}_x(z, \tau) \frac{c}{\sqrt{\epsilon}} \hat{S}_z(z, \tau), \quad (2.3.22a)$$

$$\frac{\partial}{\partial z} \hat{S}_x(z, \tau) = -\kappa \rho A \hat{J}_z(z, \tau) \hat{S}_y(z, \tau), \quad \frac{\partial}{\partial z} \hat{S}_y(z, \tau) = \kappa \rho A \hat{J}_z(z, \tau) \hat{S}_x(z, \tau). \quad (2.3.22b)$$

Next we solve these differential equations to lowest order, that is we approximate in each equation (2.3.22a) the value of the spin in any given time t with the value of the spin at the start time $t = 0$. In the equations (2.3.22b) we approximate the value of the Stokes operator at any position z with the value at the start position $z = 0$. This leads to the following equations

$$\hat{J}_x^{\text{Out}}(z) = \hat{J}_x^{\text{In}}(z) - \kappa \hat{J}_y^{\text{In}}(z) \frac{c}{\sqrt{\epsilon}} \int_0^T d\tau \hat{S}_z(z, \tau), \quad (2.3.23a)$$

$$\hat{J}_y^{\text{Out}}(z) = \hat{J}_y^{\text{In}}(z) + \kappa \hat{J}_x^{\text{In}}(z) \frac{c}{\sqrt{\epsilon}} \int_0^T d\tau \hat{S}_z(z, \tau), \quad (2.3.23b)$$

$$\hat{S}_x^{\text{Out}}(\tau) = \hat{S}_x^{\text{In}}(\tau) - \kappa \rho A \int_0^L dz' \hat{J}_z(z', \tau) \hat{S}_y^{\text{In}}(\tau), \quad (2.3.23c)$$

$$\hat{S}_y^{\text{Out}}(\tau) = \hat{S}_y^{\text{In}}(\tau) + \kappa \rho A \int_0^L dz' \hat{J}_z(z', \tau) \hat{S}_x^{\text{In}}(\tau). \quad (2.3.23d)$$

We want to treat the ensemble as one big particle of some spin, interacting with a pulse of light. Therefore we introduce the collective operators measuring the collective spin of the ensemble and

the number of photons in a laser pulse. These are given by:

$$\hat{J}_n = \rho A \int_0^L dz \hat{J}_z(z), \quad (2.3.24)$$

$$\hat{S}_n = \frac{c}{\sqrt{\epsilon}} \int_0^T d\tau \hat{S}_n(\tau). \quad (2.3.25)$$

These definitions gives rise to an additional set of commutation relations, and by calculating these we find a set similar to the single particle, and single photon commutation relations.

$$[\hat{J}_n; \hat{J}_m] = i\epsilon_{nml}\hat{J}_l, \quad \text{and} \quad [\hat{S}_n; \hat{S}_m] = i\epsilon_{nml}\hat{S}_l \quad (2.3.26)$$

Detailed calculations can be found in Appendix B.4.5. Introducing these collective operators to the equations (2.3.23) we finally arrive at:

$$\hat{J}_x^{\text{Out}} = \hat{J}_x^{\text{In}} - \kappa \hat{J}_y^{\text{In}} \hat{S}_z, \quad (2.3.27a)$$

$$\hat{J}_y^{\text{Out}} = \hat{J}_y^{\text{In}} + \kappa \hat{J}_x^{\text{In}} \hat{S}_z, \quad (2.3.27b)$$

$$\hat{S}_x^{\text{Out}} = \hat{S}_x^{\text{In}} - \kappa \hat{J}_z^{\text{In}} \hat{S}_y, \quad (2.3.27c)$$

$$\hat{S}_y^{\text{Out}} = \hat{S}_y^{\text{In}} + \kappa \hat{J}_z^{\text{In}} \hat{S}_x. \quad (2.3.27d)$$

2.4 Applications of the 1 Dimensional Theory

The equation (2.3.27) is more or less the starting point of the theory of entangled macroscopic objects. Generally the assumption making the system of equations particularly well suited for the purpose of entanglement, is to assume a strong linearly polarized incident laser pulse. This justifies treating the Stokes operator \hat{S}_x as a c-number, thus making the commutation relations on the Stokes operators appear as canonical field commutators. Define:

$$\hat{X}^p \equiv \frac{\hat{S}_y}{\sqrt{\hat{S}_x}}, \quad \hat{P}^p \equiv \frac{\hat{S}_z}{\sqrt{\hat{S}_x}} \quad (2.4.1)$$

They apply to the canonical commutation relations,

$$[\hat{X}^p; \hat{P}^p] = i. \quad (2.4.2)$$

Similarly we may assume the ensemble to be macroscopic polarized in the x -direction, we thus treat the operator \hat{J}_x as a c -number. Similar definitions,

$$\hat{X}^a \equiv \frac{\hat{J}_y}{\sqrt{J_x}}, \quad \hat{P}^a \equiv \frac{\hat{J}_z}{\sqrt{J_x}} \quad (2.4.3)$$

lead to similar canonical commutation relation

$$[\hat{X}^a, \hat{P}^a] = i. \quad (2.4.4)$$

Inserting these definitions in the equations (2.3.27-d) letting prime denote the outcome of the operator, results in the following set of equations.

$$\hat{X}^{p'} = \hat{X}^p + \kappa \hat{P}^a \quad (2.4.5)$$

$$\hat{X}^{a'} = \hat{X}^a + \kappa \hat{P}^p \quad (2.4.6)$$

Considering two macroscopically polarized atomic ensembles, of spatial separation, denoted by 1 and 2. One may define the set of EPR-like operators, $\hat{X}_1^a - \hat{X}_2^a$ and $\hat{P}_1^a + \hat{P}_2^a$. They are a set of operators commuting on the combined system, though not on each system alone.

$$[\hat{X}_1^a - \hat{X}_2^a, \hat{P}_1^a + \hat{P}_2^a] = [\hat{X}_1^a, \hat{P}_1^a] - [\hat{X}_2^a, \hat{P}_2^a] = i - i = 0 \quad (2.4.7)$$

It has been shown that two systems are said to be entangled with respect to a set of EPR-operators, if the variance of of these operators obey the following inequality(40).

$$\delta(\hat{X}_1^a - \hat{X}_2^a)^2 + \delta(\hat{P}_1^a + \hat{P}_2^a)^2 \leq 2 \quad (2.4.8)$$

The inequality can be tested experimentally (25; 29) by considering the setup shown in Figure 2.2. A linearly polarized strong laser pulse propagates successively through the two atomic samples.

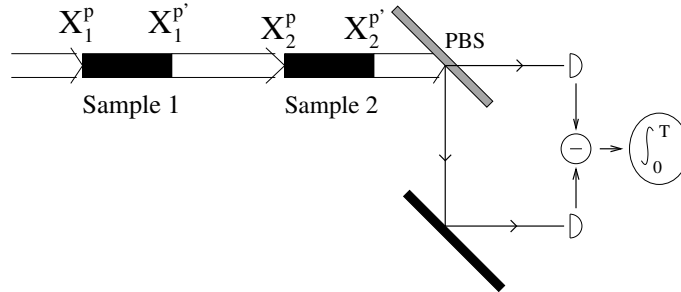


Figure 2.2: Schematic setup for Bell measurements.

The linear light may be decomposed into two circular polarization-modes (a_1, a_2) , in which case the two polarization modes appears as: $(a_1 + ia_2)/\sqrt{2}$ and $(a_1 - ia_2)/\sqrt{2}$. These two modes are split on a polarizing beam splitter (PBS), and the difference of the two photo currents, when integrated over the pulse duration, is measured. In section 7.3 we note that the Stokes operator \hat{S}_z corresponds to the difference in intensity between sigma-plus light and sigma-minus light. The Stokes operator \hat{S}_y measures the difference between the intensity of linearly polarized photons along a direction of plus 45 degrees and minus 45 degrees. The setup depicted in figure 2.2 thus measures the stokes operator \hat{S}_y , in our notation, the setup measures the operator $\hat{X}_2^{p'}$ of the laserpulse after having exited the second sample. According to our equations (2.4.5), when used for the two samples, we find the operator to be given as:

$$\hat{X}_2^{p'} = \hat{X}_1^p + \kappa(\hat{P}_1^a + \hat{P}_2^a), \quad (2.4.9)$$

where we used equation (2.4.5). This way we aquired a collective measurement of $\hat{P}_1^a + \hat{P}_2^a$, with some inherent vaccum noise. Then the collective atomic spin is rotated around the x -axis. This can be obtained with negligible noise by applying a classical laserpulse of large detuning. (25) This operation makes the following transformation of the measured variables.

$$\hat{X}_1^a \rightarrow -\hat{P}_1^a, \quad \hat{P}_1^a \rightarrow \hat{X}_1^a \quad \text{and} \quad \hat{X}_2^a \rightarrow \hat{P}_2^a, \quad \hat{P}_2^a \rightarrow -\hat{X}_2^a \quad (2.4.10)$$

This again leads to a transformation of the first measured variables, in wich case $\hat{P}_1^a + \hat{P}_2^a$ in the new basis is represented as $\hat{X}_1^a - \hat{X}_2^a$. A second measurement on the system returns $\hat{P}_1^a + \hat{P}_2^a$. This way we have measured both the EPR operators $\hat{X}_1^a - \hat{X}_2^a$ and $\hat{P}_1^a + \hat{P}_2^a$, and we have now left the combined system in an eigenstate of the two EPR-operators. The variance of the two operators satisfies the following(25):

$$\delta(\hat{X}_1^a - \hat{X}_2^a)^2 + \delta(\hat{P}_1^a + \hat{P}_2^a)^2 = \frac{2}{1 + 2\kappa^2}. \quad (2.4.11)$$

It is thus shown possible, using only coherent light, to generate continuous variable entanglement between two nonlocal ensembles. The degree of entanglement depends exclusively on the parameter κ . The experiment, in its more elaborate form, is currently taking place at NBI by Eugene Polzik *et al.*

2.5 Problems with the 1-Dimensional Theory

The essential result of the 1-dimensional theory says, that it is possible to detect quantum behavior of a macroscopic object. The degree of entanglement depends on the coupling constant κ . The

bigger κ the more entanglement. In the derivation of the founding equations (2.3.27) we neglected all scattering processes, it is therefore only natural to ask, “will the conclusions be the same taking into account scattering and decay?”

The type of scattering process where an atom absorbs a photon, emits one that gets absorbed by another atom, a process referred to as light induced dipole interactions, are up to now, unaccounted for. This process is nonexistent in the one dimensional theory. Few theoretical attempts has been made to resolve this effect in three dimensions. (41) In the article (25) attempts are made to include the decay of atomic spin, by introducing vacuum noise operators.

In the real three dimensional ensemble, on a microscopic level, photons do interact with atoms. It will happen, though with a probability depending on the detuning, that a photon interact with an atom changing the internal state of the atom. It is possible to account for these kind of processes on a single atom system,[†] the real trick is to extrapolate to a macroscopic ensemble of atoms. On the other hand we have a well tested, well functioning classical theory on diffraction with polarizable medias. We know for instance that the displaced electric field in a polarized media evolve according to the following wave equation.[‡]

$$\nabla^2 \mathbf{D}(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{D}(\mathbf{r}, t) = -\nabla \times \nabla \times \mathbf{P}(\mathbf{r}, t) \quad (2.5.1)$$

where $\mathbf{D}(\mathbf{r}, t)$ is the displaced electric field, and $\mathbf{P}(\mathbf{r}, t)$ the polarization of the media. Dealing with the amount of atoms within our ensembles, justify treating the interaction problem in frames of a continuous media. The standard quantization of the electromagnetic field is based on microscopic systems, however our system is macroscopic, leaving us with the question: To what extent can we trust our fundamental microscopic ansatz when extrapolating to a macroscopic system?

A simple calculation on the standard quantized Hamiltonian given by:[§]

$$\hat{\mathcal{H}} = \frac{1}{2} \int d^3 r \left\{ \epsilon_0 \hat{\mathbf{E}}(\mathbf{r}, t)^2 + \frac{1}{\mu_0} (\nabla \times \hat{\mathbf{A}}(\mathbf{r}, t))^2 - 2\hat{\mathbf{E}}(\mathbf{r}, t) \cdot \hat{\mathbf{P}}(\mathbf{r}, t) \right\} \quad (2.5.2)$$

will reveal some notion on the previous question. We have quantized with standard commutation relations, $[\hat{\mathbf{A}}(\mathbf{r}, t); \hat{\mathbf{E}}(\mathbf{r}', t)] = i\hbar\delta(\mathbf{r} - \mathbf{r}')$, and using Heisenberg's equation of motion we find the following wave-equation.

$$\nabla^2 \hat{\mathbf{E}}(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \hat{\mathbf{E}}(\mathbf{r}, t) = -\nabla \times \nabla \times \hat{\mathbf{P}}(\mathbf{r}, t) \quad (2.5.3)$$

[†]Consider the Wigner-Weiskopf theory of spontaneous emission.

[‡]See, e.g., J.D. Jackson (42)

[§]See, e.g., the book (36)

The obvious difference between the two equations (2.5.1) and (2.5.3) only encourage a thorough remake of the quantized theory on light matter interaction. Combined with the problems of dealing with three dimensional scattering processes, this sets the stage for this ongoing research work.

Part II

3-Dimensional System: Equations of Motion

Lagrangian formulation

3.1 Lagrangian

In this section we lay the foundation for the development of the three dimensional theory. As pointed out in section 2.5, we seek a description in complete agreement with classical theories, thus we start with the basic principles of classical mechanics. We wish to derive a Lagrangian capable of describing our system, a Lagrangian being Gauge-invariant. In the following we sum up some of the important results from classical mechanics, *Hamilton's Principle*(43).

The motion of the system from time t_1 to t_2 is such that the line integral (called the action or the action integral,

$$S = \int_{t_1}^{t_2} L dt \quad (3.1.1)$$

where $L = T - V$, has a stationary value for the actual path of the motion. T is the kinetic energy of the system and V the potential.

The functional L is called the Lagrangian. In generalized form, the Lagrangian is a functional depending on some generalized coordinates, e.g., fields in the continuous case.(44) The generalized action concerning fields on n -dimensional space is given by:

$$S[\phi] = \int d^n x \mathcal{L}(\phi_r, \partial_\mu \phi_r) \quad (3.1.2)$$

where $\mathcal{L}(\phi_r, \partial_\mu \phi_r)$ is the Lagrangian density, $\phi_r(\mathbf{x})$ is the r -th vector component of the field $\phi(\mathbf{x})$,

and $\partial_\mu \phi_r(\mathbf{x})$ is the derivative of the r -th component of the field with respect to the μ -th coordinate. We use the Einstein summation notation as noted in Notation and Convention . Assuming a set of fixed boundary conditions on the fields $\phi_r(\mathbf{x})$, e.g., they vanish at infinity, we may derive the *Euler-Lagrange equations of motion*. Derivation can be found in Appendix C.1.

$$\frac{\partial \mathcal{L}}{\partial \phi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_r)} = 0 \quad (3.1.3)$$

In principle we can derive any physical behavior of a system, given we know the Lagrangian describing it. This however introduces a problem of knowing the Lagrangian which might be a task as difficult as knowing the motional behavior of a system. Our line of approach will be firstly to base our choice of Lagrangian on a requirement of Gauge-invariance. Secondly that the expected equations of motion are derivable from Euler-Lagrange equations of motion. By expected equations of motion we mean the Maxwell Equations and Lorentz Force Law. Below is stated the *Gauge Transformation*, the transformation on which our Lagrangian shall be invariant.

$$\mathbf{A}(\mathbf{r}, t) \rightarrow \mathbf{A}'(\mathbf{r}, t) = \mathbf{A}(\mathbf{r}, t) + \nabla \psi(\mathbf{r}, t), \quad \phi(\mathbf{r}, t) \rightarrow \phi'(\mathbf{r}, t) = \phi(\mathbf{r}, t) - \frac{\partial}{\partial t} \psi(\mathbf{r}, t) \quad (3.1.4)$$

It seems a deeply fundamental principle in physics, that we may describe physical quantities in terms of nonphysical ones. This often leaves us with some symmetries in choice of the nonphysical parameters, where we may change on the nonphysical parameters, without changing the physical ones. A good example is using the vector and scalar potential to describe the physical quantities such as electric and magnetic field, see equation (3.1.5). We may make the Gauge transformation on the vector and scalar fields, and it do not change the physical electric and magnetic fields. In this respect we want our Lagrangian, the generator of equations of motion for our physical fields, to be Gauge invariant. Our physical assumptions regarding the system will be the same as in the one dimensional theory, that is we treat the atoms in the *one electron approximation* and also use the *dipole approximation*. In order not to complicate things too much we will treat the atoms as lying still, the motion of atoms introduce first Doppler broadening and second decoherence of ground state due to atomic collisions. The Doppler broadening is insignificant for an off-resonant interaction, and the atomic collisions happen on a timescale that can be considered long for many experimental purposes. (25) The system in consideration thus involves a physical field describing the spacing between nucleus and electron $\chi(\mathbf{r}, t)$, a vector potential $\mathbf{A}(\mathbf{r}, t)$ and a scalar potential $\phi(\mathbf{r}, t)$, where the physical electric- and magnetic-fields can be found from the vector- and scalar-

potential as,

$$\mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} - \nabla \phi(\mathbf{r}, t) \quad (3.1.5a)$$

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t). \quad (3.1.5b)$$

Similarly we may write out the field $\chi(\mathbf{r}, t)$, describing electron - nucleus separation for all atoms.

$$\chi(\mathbf{r}, t) = \sum_j^{\text{Atoms}} (\mathbf{r}_j^e - \mathbf{r}_j) \delta(\mathbf{r} - \mathbf{r}_j) \quad (3.1.5c)$$

The vector \mathbf{r}_j^e is the position of the electron of the j th atom, and \mathbf{r}_j the position of the j th atom. The Lagrangian density is a tensor of zeroth rank, it involves the above mentioned fields, and it is Gauge-invariant. Let us consider the following Lagrangian density.

$$\begin{aligned} \mathcal{L}(\mathbf{A}, \frac{\partial}{\partial t} \mathbf{A}, \phi, \frac{\partial}{\partial t} \phi, \chi, \frac{\partial}{\partial t} \chi) = & \frac{1}{2} \left\{ \epsilon_0 \left(-\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} - \nabla \phi(\mathbf{r}, t) \right)^2 - \frac{1}{\mu_0} \left(\nabla \times \mathbf{A}(\mathbf{r}, t) \right)^2 \right\} \\ & + g \left(-\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} - \nabla \phi(\mathbf{r}, t) \right) \cdot \chi(\mathbf{r}, t) + \frac{m}{2} \left(\frac{\partial \chi(\mathbf{r}, t)}{\partial t} \right)^2 - e U_{\text{Atom}}(\chi) \end{aligned} \quad (3.1.6)$$

The functional $U_{\text{Atom}}(\chi)$ describes the potential energy of the system, electron - nucleus. This could include the coulomb potential between nucleus and electron as well as the spin - orbit coupling. g is some coupling constant, m the mass of the electron and e the electron charge.

3.2 Classical Equations of Motion

In this section we derive the classical equations of motion from the Lagrangian in equation (3.1.6). First let us consider the atoms. Setting the Euler-Lagrange equation (3.1.3) to fit our three dimensional system, it reads:

$$\frac{\partial \mathcal{L}}{\partial \phi_r} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r} = 0 \quad (3.2.1)$$

3.2.1 Lorentz Force

Doing the functional derivatives on the Lagrangian density with respect to the field $\chi(\mathbf{r}, t)$ we will derive the following equation. For details we refer to Appendix C.2.

$$m \frac{\partial^2}{\partial t^2} \chi(\mathbf{r}, t) = -e \frac{\partial U_{\text{Atom}}(\chi)}{\partial \chi} + g \mathbf{E}(\mathbf{r}, t) \quad (3.2.2)$$

We used the shorthand notation $\frac{\partial U_{\text{Atom}}(\chi)}{\partial \chi}$ for the gradient of the potential $U_{\text{Atom}}(\chi)$ with respect to the generalized coordinate $\chi(\mathbf{r}, t)$. Considering only one atom and placing this in the center, the generalized coordinate $\chi(\mathbf{r}, t)$ reduce to the Euclidean position vector, and one may derive the usual Lorentz Force.* We also note that from equation (3.2.2) we may fix the coupling constant g . In order to represent the usual Lorentz force we must have $g = -e$.

3.2.2 Maxwell Equations

Let us consider the equations one may derive from Euler Lagrange equation of motion, considering the vector potential $\mathbf{A}(\mathbf{r}, t)$. In Appendix C.2 we put the detailed calculations concerning the result stated in equation (3.2.3) and (3.2.4).

$$\frac{\partial}{\partial t} \mathbf{D}(\mathbf{r}, t) = \frac{1}{\mu_0} \nabla \times \mathbf{B}(\mathbf{r}, t) \quad (3.2.3)$$

Considering the remaining equation we find the final Maxwell equation.

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = 0 \quad (3.2.4)$$

Comfortably assured by the good nature of our Lagrangian, we will in the next chapter proceed by quantizing our theory. Our initial requirement of being able to reproduce classical theory has thus far been met.

*See (43) pg. 22 for derivation

Hamiltonian formulation

This chapter is devoted to the task of deriving the desired Hamiltonian from our Lagrangian. We will then quantize this Hamiltonian. Bear in mind the different levels of quantizations, at the simplest level, there is the canonical quantization. More advanced quantization procedures may be better suited in some cases, but they have to correspond effectively to the canonical quantization. A way to understand the different levels of quantizations, would be to say that for this particular system the set of possible field configurations are limited. We could extract this knowledge from the otherwise arbitrary field, in which case the thus free variables are now concerning the particular kind of system in consideration. This method of extracting knowledge from an otherwise complicated set of equations, is generally used when solving problems. Should we examine the transportation of goods in this country, we could imagine a large set of complicated equations to explain this process. Since we know transportation is happening along roads this information we could therefore take for granted, and hence neglect the part of the big set of equations explaining transport, having to deal with the fact that transportation takes place along the road.

We will find that not only is it possible to extract spatial system behavior from the governing set of equations, the Hamiltonian, but we may also acquire knowledge regarding nontrivial equal-space commutationrelations, something that is not easily found considering canonical quantization procedures. The nontrivial equal-space commutationrelations are essential results that we will use in the calculations done in Part III.

4.1 Legendre transformation

The classical Hamiltonian is related to the classical Lagrangian through the following *Legendre Transformation*

$$\mathcal{H}(\mathbf{A}, \mathbf{\Pi}, \phi, \Phi, \chi, \theta) = \int d^3r \frac{\partial \mathbf{A}}{\partial t} \cdot \mathbf{\Pi} + \frac{\partial \phi}{\partial t} \Phi + \frac{\partial \chi}{\partial t} \theta - \mathcal{L}(\mathbf{A}, \frac{\partial}{\partial t} \mathbf{A}, \phi, \frac{\partial}{\partial t} \phi, \chi, \frac{\partial}{\partial t} \chi) \quad (4.1.1)$$

We introduced new fields, the canonical momenta, $\mathbf{\Pi}(\mathbf{r}, t), \Phi(\mathbf{r}, t), \theta(\mathbf{r}, t)$. These are, as vector components, given by:

$$\Pi_r = \frac{\partial \mathcal{L}}{\partial \dot{A}_r}, \quad \Phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}, \quad \theta_r = \frac{\partial \mathcal{L}}{\partial \dot{\chi}_r} \quad (4.1.2)$$

Calculating the canonical momentas, and performing the Legendre transformation described in equation (4.1.1), we find the folowing Hamiltonian. Detailed calculations are found in Appendix C.3.

$$\mathcal{H} = \int d^3r \frac{1}{2} \left(\frac{\mathbf{\Pi}^2}{\epsilon_0} + \frac{(\nabla \times \mathbf{A})^2}{\mu_0} + \frac{e^2 \chi^2}{\epsilon_0 m} \right) - \nabla \phi \cdot \mathbf{\Pi} - \frac{e}{\epsilon_0} \mathbf{\Pi} \cdot \chi + \frac{\theta^2}{2m} + eU_{\text{atom}}(\chi). \quad (4.1.3)$$

In the Hamiltonian an energy term, $\frac{e^2 \chi^2}{\epsilon_0 m}$ appeared. By simple inspection of definition (3.1.5c) one finds that this term results in an infinite contribution. In our approximation scheme a deltafunction described the atom, this had by all means been better described by a localized wavepackage. The term had then been a self-coupling term of no importance for the dynamics of our system. We do not choose this wavepackage description since treating particle density correlation simplifies when using a delta function. However based on the wavepackage argumentation we will neglect the self coupling term. Using the *Hamiltonian equations of motion* we may derive the Maxwell equations, this is a straight forward calculation and has been put in Appendix C.4.

$$\frac{\partial \mathcal{H}}{\partial P_r} = \frac{\partial X_r}{\partial t}, \quad \frac{\partial \mathcal{H}}{\partial X_r} = -\frac{\partial P_r}{\partial t} \quad (4.1.4)$$

where

$$\mathcal{H}(\mathbf{P}, \mathbf{X}) = \int d^3r \mathcal{H}(\mathbf{P}, \mathbf{X}). \quad (4.1.5)$$

\mathbf{P} being canonical momenta with respect to \mathbf{X} .

Having established the Hamiltonian describing the system, we continue by quantizing it. We

will identify the terms

$$\frac{\theta^2}{2m} + eU_{\text{atom}}(\chi) \quad (4.1.6)$$

with the energy of the atomic system, and assume known eigenfunctions for this operator. This essentially leaves us with the atomic Hamiltonian in equation (2.2.11). When considering the fields, the canonical quantization procedure is to impose standard commutation relation on the field $\mathbf{A}(\mathbf{r}, t)$ and the corresponding canonical momentum $\mathbf{\Pi}(\mathbf{r}, t)$, that is

$$[\mathbf{A}(\mathbf{r}, t); \mathbf{\Pi}(\mathbf{r}', t)] = i\hbar\bar{I}\delta(\mathbf{r} - \mathbf{r}'). \quad (4.1.7)$$

The matrix \bar{I} is the identity matrix. By foresight we prefer to consider the field in a basis in which we may deal with single photons. In the case of free space, general procedure is by clever choice of spatial basis functions* to rewrite the Hamiltonian in a form comparable to the harmonic oscillator. In this case, a photon of some mode is a representation of an excitation of an oscillator corresponding to this spatial mode in consideration. Our system is however not free, we are dealing with a media, an atomic gas. Our assumption is then that it is possible for us to extract the dispersive effect of the atomic gas, and include this in the part of our Hamiltonian that we may diagonalize. The remaining terms of our Hamiltonian we consider as perturbations. In this respect, we consider our system as photons in a dispersive media, and not vacuum, interacting weakly with some perturbation. This introduces the problem of quantizing the electromagnetic field in a dispersive media. In the following section we therefore search a quantized expansion of the electromagnetic fields, similar to equation (2.2.5), that will diagonalize the part of our Hamiltonian describing a dispersive media.

4.2 General Space-Time Evolution of Fields in Matter

When considering quantization of the displaced electric field, we expect to find some spatial mode functions describing the propagation of light through the media. We know, from ray optics, the classical evolution of light through a media, this same structure we expect to find in a quantized theory as well. In this derivation we follow a path first laid by Glauber and Lewenstein (45) in 1991. Since the work on quantizing the Hamiltonian is closely related to the calculations done by Glauber and Lewenstein, we will put most of the calculations in Appendix C, and focus mainly on the methods as well as the results. Let us walk a few steps back, and reconsider the Lagrangian given in equation (3.1.6). Let us assume that the field describing polarization of the atoms $\chi(\mathbf{r}, t)$

*This is typically plane waves see, e.g., (4; 36)

is in fact linearly depending on the electric field, that is

$$\chi(\mathbf{r}, t) = p(\mathbf{r}) \left(-\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} - \nabla \phi(\mathbf{r}, t) \right) \quad (4.2.1)$$

where the function $p(\mathbf{r})$ is only depending on space and not time. Let us define the function

$$\epsilon(\mathbf{r}) \equiv 1 + \frac{-2ep(\mathbf{r})}{\epsilon_0}. \quad (4.2.2)$$

In these calculations we consider only dispersive media, e.g., a piece of plastic. Using these assumptions and definitions we may rewrite the Lagrangian density (3.1.6) to the following.

$$\mathcal{L}(\mathbf{A}, \frac{\partial}{\partial t} \mathbf{A}, \phi, \frac{\partial}{\partial t} \phi, \chi, \frac{\partial}{\partial t} \chi) = \frac{1}{2} \left\{ \epsilon_0 \epsilon(\mathbf{r}) \left(-\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} - \nabla \phi(\mathbf{r}, t) \right)^2 - \frac{1}{\mu_0} (\nabla \times \mathbf{A}(\mathbf{r}, t))^2 \right\} \quad (4.2.3)$$

Calculating the canonical momentum for the field $\mathbf{A}(\mathbf{r}, t)$ we find,

$$\mathbf{\Pi}(\mathbf{r}, t) = -\epsilon_0 \epsilon(\mathbf{r}) \left(-\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} - \nabla \phi(\mathbf{r}, t) \right) \quad \text{or} \quad \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} = -\nabla \phi(\mathbf{r}, t) + \epsilon^{-1}(\mathbf{r}) \frac{\mathbf{\Pi}(\mathbf{r}, t)}{\epsilon_0}. \quad (4.2.4)$$

Using this along with equation (4.1.1), and choosing a Gauge where $\nabla \phi(\mathbf{r}, t) = 0$, we find a Hamiltonian given by:

$$\mathcal{H} = \int d^3r \frac{1}{2} \left\{ \frac{\mathbf{\Pi}(\mathbf{r}, t)^2}{\epsilon_0 \epsilon(\mathbf{r})} + \frac{1}{\mu_0} (\nabla \times \mathbf{A}(\mathbf{r}, t))^2 \right\} \quad (4.2.5)$$

Naively quantizing the Hamiltonian using commutation relations

$$[\hat{\mathbf{A}}(\mathbf{r}, t); \mathbf{\Pi}(\mathbf{r}', t)] = i\hbar \delta_{\mathbf{T}}(\mathbf{r}, \mathbf{r}'), \quad (4.2.6)$$

we may find, using Heisenberg equations of motion, the Maxwell equations given by[†]:

$$\frac{\partial}{\partial t} \hat{\mathbf{D}}(\mathbf{r}, t) = \frac{1}{\mu_0} \nabla \times \hat{\mathbf{B}}(\mathbf{r}, t) \quad (4.2.7a)$$

$$\frac{\partial}{\partial t} \hat{\mathbf{B}}(\mathbf{r}, t) = -\frac{1}{\epsilon_0} \nabla \times \frac{\hat{\mathbf{B}}(\mathbf{r}, t)}{\epsilon(\mathbf{r})} \quad (4.2.7b)$$

where we identified the displaced electric field, and the magnetic field. These may be combined in the following wave equation concerning the displaced electric field.

$$\frac{\partial^2}{\partial t^2} \hat{\mathbf{D}}(\mathbf{r}, t) + c^2 \nabla \times \nabla \times \epsilon^{-1}(\mathbf{r}) \hat{\mathbf{D}}(\mathbf{r}, t) = 0 \quad (4.2.8)$$

[†]We will not go in detail with these calculations since they are done extensively in chapter 5.

This equation tells us how an unperturbed displaced electric field propagate through a dispersive media, thus the spatial mode functions in which we in the following expand our Hamiltonian, must follow this general evolution. Our next concern is therefore to find appropriate mode functions, and then diagonalize the Hamiltonian.

4.2.1 Diagonalizing the Hamiltonian

Let us consider the set of eigenmode equations given by

$$\nabla \times \nabla \times \frac{\mathbf{f}_k(\mathbf{r})}{\epsilon(\mathbf{r})} = \frac{\omega_k^2}{c^2} \mathbf{f}_k(\mathbf{r}) \quad (4.2.9a)$$

$$\nabla \cdot \mathbf{f}_k(\mathbf{r}) = 0 \quad (4.2.9b)$$

This set of defining equation for basis functions is chosen with some hindsight,[‡] it makes this description of light matter interaction simpler. As with many problems the ability to find a solution often depends on the frame in which one search the solution. We will in the following show that these functions defined in equation (4.2.9) do in fact span the set of transverse functions. To do that we define a set of vector functions $\{\mathbf{g}_k(\mathbf{r})\}$ defined by:

$$\mathbf{g}_k(\mathbf{r}) = \frac{\mathbf{f}_k(\mathbf{r})}{\sqrt{\epsilon(\mathbf{r})}} \quad (4.2.10)$$

This we use in equation (4.2.9a) and find the eigenmode equation

$$\frac{1}{\sqrt{\epsilon(\mathbf{r})}} \nabla \times \nabla \times \frac{1}{\sqrt{\epsilon(\mathbf{r})}} \mathbf{g}_k(\mathbf{r}) = \frac{\omega_k^2}{c^2} \mathbf{g}_k(\mathbf{r}) \quad (4.2.11a)$$

$$\nabla \cdot \sqrt{\epsilon(\mathbf{r})} \mathbf{g}_k(\mathbf{r}) = 0 \quad (4.2.11b)$$

We may show that the differential operator $D[\cdot]$ given by $D[\cdot] = \frac{1}{\sqrt{\epsilon(\mathbf{r})}} \nabla \times \nabla \times \frac{1}{\sqrt{\epsilon(\mathbf{r})}}$ is hermitian. Given two arbitrary vector functions $\phi(\mathbf{r}, t)$ and $\psi(\mathbf{r}, t)$ we define the inner product to be the usual,

$$\langle \phi(\mathbf{r}, t) | \psi(\mathbf{r}, t) \rangle = \int d^3r \phi(\mathbf{r}, t) \cdot \psi^*(\mathbf{r}, t). \quad (4.2.12)$$

Calculating the inner product, $\langle \phi(\mathbf{r}, t) | D[\psi(\mathbf{r}, t)] \rangle$ we find the differential operator $D[\cdot]$ to be hermitian and the set $\{\mathbf{g}_k(\mathbf{r})\}$ may be chosen as a basis for the space of L_2 functions subject to requirement (4.2.11b). Detailed calculations can be found in Appendix C.5 The orthonormality and

[‡]Trial and error, leading to nicest results.

completeness relations reads

$$\bar{\bar{\delta}}_\epsilon(\mathbf{r}, \mathbf{r}') = \sum_{\mathbf{k}} \mathbf{g}_{\mathbf{k}}(\mathbf{r}) \mathbf{g}_{\mathbf{k}}^*(\mathbf{r}') \quad (4.2.13a)$$

$$\delta_{\mathbf{k}\mathbf{k}'} = \int d^3r \mathbf{g}_{\mathbf{k}}(\mathbf{r}) \cdot \mathbf{g}_{\mathbf{k}'}^*(\mathbf{r}) \quad (4.2.13b)$$

where we introduced the distribution $\bar{\bar{\delta}}_\epsilon(\mathbf{r}, \mathbf{r}')$ that is the generalized transverse delta function. It reduces to the standard transverse delta function when $\epsilon(\mathbf{r}) = 1$. Consider an arbitrary field $\mathbf{Y}_T(\mathbf{r}, t)$ submitting to the following:

$$\nabla \cdot \sqrt{\epsilon(\mathbf{r})} \mathbf{Y}_T(\mathbf{r}, t) = 0 \quad (4.2.14)$$

In that case we may expand this field in the basis $\{\mathbf{g}_{\mathbf{k}}(\mathbf{r})\}$. The effect of the generalized transverse delta function thus reads:

$$\begin{aligned} \int d^3r' \bar{\bar{\delta}}_\epsilon(\mathbf{r}, \mathbf{r}') Y_T(\mathbf{r}', t) &= \sum_{\mathbf{k}\mathbf{k}'} \int d^3r' \mathbf{g}_{\mathbf{k}}(\mathbf{r}) \mathbf{g}_{\mathbf{k}'}^*(\mathbf{r}') \cdot Y(t)_{\mathbf{k}'} \mathbf{g}_{\mathbf{k}'}(\mathbf{r}') \\ &= \sum_{\mathbf{k}\mathbf{k}'} \mathbf{g}_{\mathbf{k}}(\mathbf{r}) Y(t)_{\mathbf{k}'} \delta_{\mathbf{k}\mathbf{k}'} = Y_T(\mathbf{r}', t) \end{aligned} \quad (4.2.15)$$

Since the transformation from the functions $\{\mathbf{g}_{\mathbf{k}}(\mathbf{r})\}$ to $\{\mathbf{f}_{\mathbf{k}}(\mathbf{r})\}$ is nothing more than a linear translation, the set $\{\mathbf{f}_{\mathbf{k}}(\mathbf{r})\}$ spans the space of transverse L_2 functions where we used the requirement (4.2.9b). Regarding orthonormality of the set $\{\mathbf{f}_{\mathbf{k}}(\mathbf{r})\}$ we find the following useful identity:

$$\begin{aligned} \delta_{\mathbf{k}\mathbf{k}'} &= \int d^3r \mathbf{g}_{\mathbf{k}}(\mathbf{r}) \cdot \mathbf{g}_{\mathbf{k}'}^*(\mathbf{r}) \\ &= \int d^3r \frac{1}{\sqrt{\epsilon(\mathbf{r})}} \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \cdot \frac{1}{\sqrt{\epsilon(\mathbf{r})}} \mathbf{f}_{\mathbf{k}'}^*(\mathbf{r}) \\ &= \int d^3r \frac{1}{\epsilon(\mathbf{r})} \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \cdot \mathbf{f}_{\mathbf{k}'}^*(\mathbf{r}). \end{aligned} \quad (4.2.16)$$

The natural normalized inner product on the space spanned by the functions $\{\mathbf{f}_{\mathbf{k}}(\mathbf{r})\}$ thus reads:

$$\langle \phi(\mathbf{r}, t) | \psi(\mathbf{r}, t) \rangle = \int d^3r \frac{1}{\epsilon(\mathbf{r})} \phi(\mathbf{r}, t) \cdot \psi^*(\mathbf{r}, t) \quad (4.2.17)$$

The distribution (4.2.13a) is an identity on the subspace of functions subject to requirement (4.2.11b), let us therefore define the analog distribution,

$$\bar{\delta}^\top(\mathbf{r}, \mathbf{r}') = \sum_{\mathbf{k}} \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \mathbf{f}_{\mathbf{k}}^*(\mathbf{r}') \quad (4.2.18)$$

This distribution is the transverse delta function, on a space equipped with the inner product defined in equation (4.2.17). For an arbitrary transverse field $\mathbf{X}^T(\mathbf{r}, t)$ we may calculate

$$\begin{aligned} \int d^3 r' \frac{1}{\epsilon(\mathbf{r}')} \bar{\delta}^T(\mathbf{r}, \mathbf{r}') \mathbf{X}^T(\mathbf{r}', t) &= \int d^3 r' \frac{1}{\epsilon(\mathbf{r}')} \sum_{\mathbf{k}} \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \mathbf{f}_{\mathbf{k}}^*(\mathbf{r}') \cdot \mathbf{X}^T(\mathbf{r}', t) \\ &= \sum_{\mathbf{k}} \left(\int d^3 r' \frac{1}{\epsilon(\mathbf{r}')} \mathbf{f}_{\mathbf{k}}^*(\mathbf{r}') \cdot \mathbf{X}^T(\mathbf{r}', t) \right) \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \end{aligned}$$

This is the expansion of the field $\mathbf{X}^T(\mathbf{r}, t)$ in the basis $\{\mathbf{f}_{\mathbf{k}}(\mathbf{r})\}$, and we have found.

$$\int d^3 r' \frac{1}{\epsilon(\mathbf{r}')} \bar{\delta}^T(\mathbf{r}, \mathbf{r}') \mathbf{X}^T(\mathbf{r}', t) = \mathbf{X}^T(\mathbf{r}, t) \quad (4.2.19)$$

Let us expand the field $\mathbf{\Pi}(\mathbf{r}, t)$ in the complex conjugate basis defined by equation (4.2.9).

$$\mathbf{\Pi}(\mathbf{r}, t) = \sum_{\mathbf{k}} \sqrt{\epsilon_0} P_{\mathbf{k}}(t) \mathbf{f}_{\mathbf{k}}^*(\mathbf{r}) \quad (4.2.20)$$

Similarly we expand the field $\mathbf{A}(\mathbf{r}, t)$

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}} c \sqrt{\mu_0} Q_{\mathbf{k}}(t) \frac{\mathbf{f}_{\mathbf{k}}(\mathbf{r})}{\epsilon(\mathbf{r})} \quad (4.2.21)$$

From the reality condition on the physical field $(\mathbf{\Pi}(\mathbf{r}, t))^\dagger = \mathbf{\Pi}(\mathbf{r}, t)$ we find the following

$$\sum_{\mathbf{k}} \sqrt{\epsilon_0} P_{\mathbf{k}}(t) \mathbf{f}_{\mathbf{k}}^*(\mathbf{r}) = \sum_{\mathbf{k}} \sqrt{\epsilon_0} P_{\mathbf{k}}^\dagger(t) \mathbf{f}_{\mathbf{k}}(\mathbf{r})$$

Taking inner product on both sides with $\frac{\mathbf{f}_{\mathbf{k}'}(\mathbf{r})}{\epsilon(\mathbf{r})}$ and integrating over all space we find,

$$\begin{aligned} \int d^3 r P_{\mathbf{k}}(t) \mathbf{f}_{\mathbf{k}}^*(\mathbf{r}) \cdot \mathbf{f}_{\mathbf{k}'}(\mathbf{r}) \frac{1}{\epsilon(\mathbf{r})} &= \int d^3 r \sum_{\mathbf{k}} P_{\mathbf{k}}^\dagger(t) \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \cdot \mathbf{f}_{\mathbf{k}'}(\mathbf{r}) \frac{1}{\epsilon(\mathbf{r})} \\ P_{\mathbf{k}'}(t) &= \sum_{\mathbf{k}} P_{\mathbf{k}}^\dagger(t) \int d^3 r \frac{1}{\epsilon(\mathbf{r})} \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \cdot \mathbf{f}_{\mathbf{k}'}(\mathbf{r}) \\ P_{\mathbf{k}'}(t) &= \sum_{\mathbf{k}} P_{\mathbf{k}}^\dagger(t) U_{\mathbf{k}\mathbf{k}'} \end{aligned} \quad (4.2.22)$$

where we defined the matrix U

$$U_{\mathbf{k}\mathbf{k}'} = \int d^3 r \frac{1}{\epsilon(\mathbf{r})} \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \cdot \mathbf{f}_{\mathbf{k}'}(\mathbf{r}). \quad (4.2.23)$$

The matrix U is both symmetric and unitary, that is

$$U_{\mathbf{k}\mathbf{k}'} = U_{\mathbf{k}'\mathbf{k}}, \quad (4.2.24)$$

which follows trivially from definition (4.2.23), and

$$\sum_{\mathbf{k}'} U_{\mathbf{k}\mathbf{k}'} U_{\mathbf{k}''\mathbf{k}'}^* = \delta_{\mathbf{k}\mathbf{k}''}. \quad (4.2.25)$$

The calculation showing unitarity has been put in Appendix C.5. We note, from equation (4.2.16) we see that the matrix elements $U_{\mathbf{k}\mathbf{k}'}^*$ are the expansion coefficients when writing the function $\mathbf{f}_{\mathbf{k}}^*(\mathbf{r})$ in the basis $\{\mathbf{f}_{\mathbf{k}}(\mathbf{r}, t)\}$

$$\mathbf{f}_{\mathbf{k}}^*(\mathbf{r}) = \sum_{\mathbf{k}'} U_{\mathbf{k}\mathbf{k}'}^* \mathbf{f}_{\mathbf{k}'}(\mathbf{r}) \quad (4.2.26)$$

Having established the needed information on the basis functions we continue by bringing our Hamiltonian on diagonal quadratic form. Using partial integration, the equations (4.2.20), (4.2.21) and (4.2.16), the Hamiltonian may be written as:

$$\mathcal{H} = \frac{1}{2} \sum_{\mathbf{k}} \{P_{\mathbf{k}}^\dagger(t) P_{\mathbf{k}}(t) + \omega_{\mathbf{k}}^2 Q_{\mathbf{k}}^\dagger(t) Q_{\mathbf{k}}(t)\} \quad (4.2.27)$$

The detailed calculation can be found in Appendix C.5.

4.2.2 Quantizing the Hamiltonian

We define the equal time commutation relations for the operators $P_{\mathbf{k}}(t)$ and $Q_{\mathbf{k}}(t)$

$$[Q_{\mathbf{k}}(t); Q_{\mathbf{k}'}(t)] = [Q_{\mathbf{k}}^\dagger(t); Q_{\mathbf{k}'}^\dagger(t)] = [Q_{\mathbf{k}}(t); Q_{\mathbf{k}'}^\dagger(t)] = 0 \quad (4.2.28a)$$

$$[P_{\mathbf{k}}(t); P_{\mathbf{k}'}(t)] = [P_{\mathbf{k}}^\dagger(t); P_{\mathbf{k}'}^\dagger(t)] = [P_{\mathbf{k}}(t); P_{\mathbf{k}'}^\dagger(t)] = 0 \quad (4.2.28b)$$

$$[Q_{\mathbf{k}}(t); P_{\mathbf{k}'}(t)] = i\hbar \delta_{\mathbf{k}\mathbf{k}'} \quad (4.2.28c)$$

From equation (4.2.22) we may find the last commutation relation:

$$\begin{aligned} [Q_{\mathbf{k}}(t); P_{\mathbf{k}'}^\dagger(t)] &= [Q_{\mathbf{k}}(t); \sum_{\mathbf{k}''} P_{\mathbf{k}''}(t) U_{\mathbf{k}''\mathbf{k}'}] = \sum_{\mathbf{k}''} U_{\mathbf{k}''\mathbf{k}'} i\hbar \delta_{\mathbf{k}\mathbf{k}''} \\ &= i\hbar U_{\mathbf{k}\mathbf{k}'} \end{aligned} \quad (4.2.28d)$$

Using expansion (4.2.21) and (4.2.20) we derive the canonical commutation relation equivalent to the one written in equation (4.1.7).

$$\begin{aligned}
[\mathbf{A}(\mathbf{r}, t); \mathbf{\Pi}(\mathbf{r}', t)] &= \left[\sum_{\mathbf{k}} c \sqrt{\mu_0} Q_{\mathbf{k}}(t) \frac{\mathbf{f}_{\mathbf{k}}(\mathbf{r})}{\epsilon(\mathbf{r})}; \sum_{\mathbf{k}'} \sqrt{\epsilon_0} P_{\mathbf{k}'}(t) \mathbf{f}_{\mathbf{k}'}^*(\mathbf{r}') \right] \\
&= \sum_{\mathbf{k}\mathbf{k}'} \frac{1}{\epsilon(\mathbf{r})} \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \mathbf{f}_{\mathbf{k}'}^*(\mathbf{r}') [Q_{\mathbf{k}}(t); P_{\mathbf{k}'}(t)] \\
&= i\hbar \frac{1}{\epsilon(\mathbf{r})} \sum_{\mathbf{k}} \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \mathbf{f}_{\mathbf{k}'}^*(\mathbf{r}') \\
&= i\hbar \frac{1}{\epsilon(\mathbf{r})} \delta^{\mathbf{T}}(\mathbf{r}, \mathbf{r}') \tag{4.2.29}
\end{aligned}$$

The distribution $\delta^{\mathbf{T}}(\mathbf{r}, \mathbf{r}')$ is the one defined in equation (4.2.18). The next step will be to represent the commutation relations (4.2.28) in terms of creation and annihilation operators, fulfilling the commutation relations given by

$$[\hat{a}_{\mathbf{k}}(t); \hat{a}_{\mathbf{k}'}(t)] = 0 \tag{4.2.30a}$$

$$[\hat{a}_{\mathbf{k}}(t); \hat{a}_{\mathbf{k}'}^{\dagger}(t)] = \delta_{\mathbf{k}\mathbf{k}'} \tag{4.2.30b}$$

Assuming that both $Q_{\mathbf{k}}(t)$ and $P_{\mathbf{k}}(t)$ can be represented by linear combination of creation and annihilation operators, we consider the following relations.

$$Q_{\mathbf{k}}(t) = \sqrt{\frac{\hbar}{2\omega_{\mathbf{k}}}} \left\{ \hat{a}_{\mathbf{k}}(t) + \sum_{\mathbf{k}'} U_{\mathbf{k}\mathbf{k}'}^* \hat{a}_{\mathbf{k}'}^{\dagger}(t) \right\} \tag{4.2.31a}$$

$$P_{\mathbf{k}}(t) = i \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2}} \left\{ \hat{a}_{\mathbf{k}}^{\dagger}(t) - \sum_{\mathbf{k}'} U_{\mathbf{k}\mathbf{k}'} \hat{a}_{\mathbf{k}'}(t) \right\} \tag{4.2.31b}$$

That they submit to relation (4.2.28) one easily checks, e.g.,

$$\begin{aligned}
[Q_{\mathbf{k}}(t); P_{\mathbf{k}'}(t)] &= \left[\sqrt{\frac{\hbar}{2\omega_{\mathbf{k}}}} \left\{ \hat{a}_{\mathbf{k}}(t) + \sum_{\mathbf{k}''} U_{\mathbf{k}\mathbf{k}''}^* \hat{a}_{\mathbf{k}''}^{\dagger}(t) \right\}; i \sqrt{\frac{\hbar\omega_{\mathbf{k}'}}{2}} \left\{ \hat{a}_{\mathbf{k}'}^{\dagger}(t) - \sum_{\mathbf{k}'''} U_{\mathbf{k}'\mathbf{k}'''} \hat{a}_{\mathbf{k}'''}(t) \right\} \right] \\
&= i\hbar \frac{1}{2} \sqrt{\frac{\omega_{\mathbf{k}'}}{\omega_{\mathbf{k}}}} \left\{ [\hat{a}_{\mathbf{k}}(t); \hat{a}_{\mathbf{k}'}^{\dagger}(t)] - \sum_{\mathbf{k}''\mathbf{k}'''} U_{\mathbf{k}\mathbf{k}''}^* U_{\mathbf{k}'\mathbf{k}'''} [\hat{a}_{\mathbf{k}''}^{\dagger}(t); \hat{a}_{\mathbf{k}'''}(t)] \right\} \\
&= i\hbar \frac{1}{2} \sqrt{\frac{\omega_{\mathbf{k}'}}{\omega_{\mathbf{k}}}} \left\{ \delta_{\mathbf{k}\mathbf{k}'} + \sum_{\mathbf{k}''\mathbf{k}'''} U_{\mathbf{k}\mathbf{k}''}^* U_{\mathbf{k}'\mathbf{k}'''} \delta_{\mathbf{k}''\mathbf{k}'''} \right\} \\
&= i\hbar \frac{1}{2} \sqrt{\frac{\omega_{\mathbf{k}'}}{\omega_{\mathbf{k}}}} \left\{ \delta_{\mathbf{k}\mathbf{k}'} + \sum_{\mathbf{k}''} U_{\mathbf{k}\mathbf{k}''}^* U_{\mathbf{k}'\mathbf{k}''} \right\}.
\end{aligned}$$

Using equation (4.2.25) we find

$$=i\hbar\frac{1}{2}\{\delta_{\mathbf{k}\mathbf{k}'} + \delta_{\mathbf{k}\mathbf{k}'}\} = i\hbar\delta_{\mathbf{k}\mathbf{k}'}. \quad (4.2.32)$$

Similarly, one may check the remaining relations. Let us introduce the definitions (4.2.31) into our Hamiltonian (4.2.27). To shorten the notation we suppress the time dependence. Using properties regarding the unitary operator U , The hamiltonian reduce to the following. Detailed calculations can be found in Appendix C.5

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} \{\hat{a}_{\mathbf{k}}^{\dagger}\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{k}}^{\dagger}\} \\ &= \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} \{\hat{a}_{\mathbf{k}}^{\dagger}\hat{a}_{\mathbf{k}} + C[\epsilon]\} \end{aligned} \quad (4.2.33)$$

The constant $C[\epsilon]$ depends on the function $\epsilon(\mathbf{r})$ and is formally infinite. We thus arrived with the well known Hamiltonian. In the following section we find explicit expressions for the quantized fields $\hat{\mathbf{A}}(\mathbf{r}, t)$ and $\hat{\mathbf{H}}(\mathbf{r}, t)$.

4.2.3 Finding explicit expressions for the fields

We find the field $\hat{\mathbf{A}}(\mathbf{r}, t)$ given in terms of creation and annihilation operators by inserting equation (4.2.31) into expansion (4.2.21). The quantized vector potential expanded in creation and annihilation operators may thus be written as

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}} \frac{c}{\epsilon(\mathbf{r})} \sqrt{\frac{\hbar\mu_0}{2\omega_{\mathbf{k}}}} \{\hat{a}_{\mathbf{k}}(t)\mathbf{f}_{\mathbf{k}}(\mathbf{r}) + \hat{a}_{\mathbf{k}}^{\dagger}(t)\mathbf{f}_{\mathbf{k}}^*(\mathbf{r})\} \quad (4.2.34)$$

Similarly we find by inserting (4.2.31) into expansion (4.2.20),

$$\hat{\mathbf{H}}(\mathbf{r}, t) = -i \sum_{\mathbf{k}} \sqrt{\frac{\hbar\omega_{\mathbf{k}}\epsilon_0}{2}} \{\hat{a}_{\mathbf{k}}(t)\mathbf{f}_{\mathbf{k}}(\mathbf{r}) - \hat{a}_{\mathbf{k}}^{\dagger}(t)\mathbf{f}_{\mathbf{k}}^*(\mathbf{r})\} \quad (4.2.35)$$

The time evolution of the creation and annihilation operators can be found using *Heisenberg equation of motion* (2.2.13) and the Hamiltonian given in equation (4.2.33).

$$\hat{a}_{\mathbf{k}}(t) = \hat{a}_{\mathbf{k}}(t_0)e^{-i\omega_{\mathbf{k}}(t-t_0)}, \quad (4.2.36)$$

$$\hat{a}_{\mathbf{k}}^{\dagger}(t) = \hat{a}_{\mathbf{k}}^{\dagger}(t_0)e^{i\omega_{\mathbf{k}}(t-t_0)}. \quad (4.2.37)$$

In this respect, assuming $t_0 = 0$, the fields may be written:

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}^-(\mathbf{r}, t) + \mathbf{A}^+(\mathbf{r}, t) \quad (4.2.38a)$$

where

$$\mathbf{A}^-(\mathbf{r}, t) = \sum_{\mathbf{k}} \frac{c}{\epsilon(\mathbf{r})} \sqrt{\frac{\hbar\mu_0}{2\omega_{\mathbf{k}}}} \hat{a}_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}t} \mathbf{f}_{\mathbf{k}}(\mathbf{r}), \quad (4.2.38b)$$

$$\mathbf{A}^+(\mathbf{r}, t) = \sum_{\mathbf{k}} \frac{c}{\epsilon(\mathbf{r})} \sqrt{\frac{\hbar\mu_0}{2\omega_{\mathbf{k}}}} \hat{a}_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}t} \mathbf{f}_{\mathbf{k}}^*(\mathbf{r}). \quad (4.2.38c)$$

Similarly we write out the quantized canonical conjugate field, recognizing the field $\hat{\mathbf{\Pi}}(\mathbf{r}, t)$ being the displaced electric field as found in equation (C.3.1a).

$$\hat{\mathbf{D}}(\mathbf{r}, t) = \hat{\mathbf{D}}^-(\mathbf{r}, t) + \hat{\mathbf{D}}^+(\mathbf{r}, t) \quad (4.2.39a)$$

where

$$\hat{\mathbf{D}}^-(\mathbf{r}, t) = i \sum_{\mathbf{k}} \sqrt{\frac{\hbar\omega_{\mathbf{k}}\epsilon_0}{2}} \hat{a}_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}t} \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \quad (4.2.39b)$$

$$\hat{\mathbf{D}}^+(\mathbf{r}, t) = -i \sum_{\mathbf{k}} \sqrt{\frac{\hbar\omega_{\mathbf{k}}\epsilon_0}{2}} \hat{a}_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}t} \mathbf{f}_{\mathbf{k}}^*(\mathbf{r}) \quad (4.2.39c)$$

When considering a typical laserfield, assuming the spatial evolution primarily follows some mode $\mathbf{f}_{\mathbf{k}_L}(\mathbf{r})$, we see that we may extract a fast oscillating part of the field. Letting tilde denote slow oscillating operators we have:

$$\hat{\mathbf{D}}(\mathbf{r}, t) = \tilde{\mathbf{D}}^-(\mathbf{r}, t) e^{-i\omega_L t} + \tilde{\mathbf{D}}^+(\mathbf{r}, t) e^{i\omega_L t}. \quad (4.2.40)$$

When again considering the Hamiltonian describing the our system (4.1.3), we will treat all terms differing from dispersion as a perturbation, these calculations thus being a good starting point for perturbation theory. We end the story on the basisfunctions $\mathbf{f}_{\mathbf{k}}(\mathbf{r})$ by calculating different commutation relations that will come in handy at a late stage.

4.2.4 Commutation Relations

From equation (4.2.38) and (4.2.39) along with commutation relation (4.2.30) one easily finds the first relation:

$$[\hat{\mathbf{A}}^+(\mathbf{r}, t); \hat{\mathbf{D}}^+(\mathbf{r}', t)] = [\hat{\mathbf{A}}^-(\mathbf{r}, t); \hat{\mathbf{D}}^-(\mathbf{r}', t)] = 0 \quad (4.2.41)$$

Let us therefore calculate:

$$\begin{aligned} [\hat{\mathbf{A}}^+(\mathbf{r}, t); \hat{\mathbf{D}}^-(\mathbf{r}', t)] &= \left[\sum_{\mathbf{k}} \frac{c}{\epsilon(\mathbf{r})} \sqrt{\frac{\hbar\mu_0}{2\omega_{\mathbf{k}}}} \hat{a}_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}t} \mathbf{f}_{\mathbf{k}}^*(\mathbf{r}); i \sum_{\mathbf{k}'} \sqrt{\frac{\hbar\omega_{\mathbf{k}'}\epsilon_0}{2}} \hat{a}_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}t} \mathbf{f}_{\mathbf{k}'}(\mathbf{r}') \right] \\ &= - \sum_{\mathbf{k}\mathbf{k}'} \frac{i\hbar c \sqrt{\epsilon_0\mu_0}}{2 \epsilon(\mathbf{r})} e^{i\omega_{\mathbf{k}}t} e^{-i\omega_{\mathbf{k}'}t} \mathbf{f}_{\mathbf{k}}^*(\mathbf{r}) \mathbf{f}_{\mathbf{k}'}(\mathbf{r}') \delta_{\mathbf{k}\mathbf{k}'} \\ &= - \frac{i\hbar}{2} \frac{1}{\epsilon(\mathbf{r})} \sum_{\mathbf{k}} \mathbf{f}_{\mathbf{k}}^*(\mathbf{r}) \mathbf{f}_{\mathbf{k}}(\mathbf{r}') \\ &= - \frac{i\hbar}{2} \frac{1}{\epsilon(\mathbf{r})} \bar{\delta}^T(\mathbf{r}, \mathbf{r}') \end{aligned} \quad (4.2.42)$$

Our transverse deltafunction $\bar{\delta}^T(\mathbf{r}, \mathbf{r}')$ always requires a factor of $\epsilon(\mathbf{r})$ to be the usual transverse deltafunction. This factor appears neatly in the commutation relations, the only place we are going to meet it, hence to shorten notation we redefine:

$$\frac{\bar{\delta}^T(\mathbf{r}, \mathbf{r}')}{\epsilon(\mathbf{r})} \rightarrow \bar{\delta}^T(\mathbf{r}, \mathbf{r}') \quad (4.2.43)$$

In later calculations we will need to know some equal-space commutation relations regarding the quantized displaced electric field. To calculate these we will examine the behavior of the basis functions $\mathbf{f}_{\mathbf{k}}(\mathbf{r})$ on an infinitely small ball around \mathbf{r} . We therefore return to the defining equation (4.2.9). Doing our calculations in an infinitely small ball, we may safely set $\epsilon(\mathbf{r})$ constant, and the equations (4.2.9) reduce to

$$\nabla^2 \mathbf{f}_{\mathbf{k}}(\mathbf{r}) = -\frac{\epsilon(\mathbf{r})\omega_{\mathbf{k}}^2}{c^2} \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \quad (4.2.44)$$

$$\nabla \cdot \mathbf{f}_{\mathbf{k}}(\mathbf{r}) = 0 \quad (4.2.45)$$

From equation (4.2.44) we find the structure of $\mathbf{f}_{\mathbf{k}}(\mathbf{r})$ to be:

$$\mathbf{f}_{\mathbf{k}}(\mathbf{r}) = \mathbf{V}_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{r}) \cdot \mathbf{r}}, \quad (4.2.46)$$

where the following relation apply

$$\mathbf{k}(\mathbf{r})^2 = \frac{\epsilon(\mathbf{r})\omega_{\mathbf{k}}^2}{c^2} \quad (4.2.47)$$

Equation (4.2.45) reveals a degeneracy in the mode number \mathbf{k} , we thus have two different polarizations, see Figure 4.1. This result is, not surprisingly, similar to the standard plane wave solution.

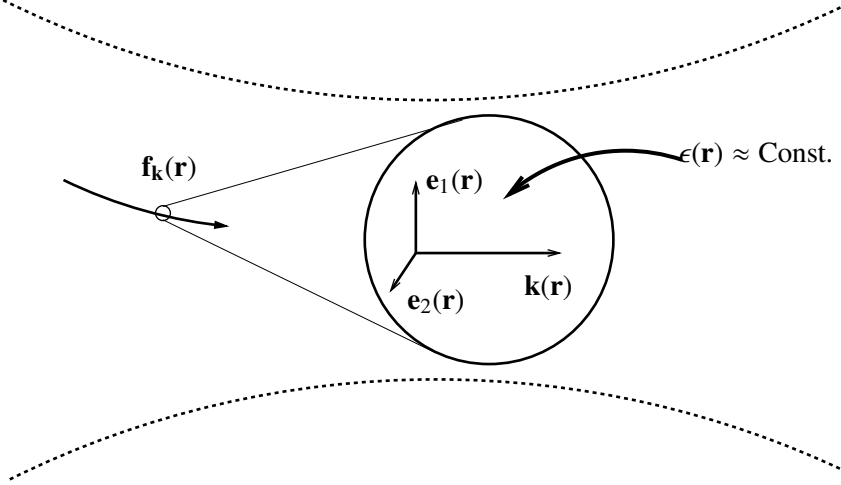


Figure 4.1: Assuming $\epsilon(\mathbf{r})$ does not change on a small enough scale, we may assume plane wave solution. The diagram shows the basis vectors in which the mode function $\mathbf{f}_{\mathbf{k}}(\mathbf{r})$ may be expanded on the small scale.

On a small enough scale our derivations are exactly similar to free space, though with a reduced light velocity. We therefore have:

$$\mathbf{f}_{\mathbf{k}}(\mathbf{r}) = \sqrt{\frac{\epsilon(\mathbf{r})}{(2\pi)^3}} \sum_s \mathbf{e}_{\mathbf{k}s}(\mathbf{r}) e^{i\mathbf{k}(\mathbf{r}) \cdot \mathbf{r}} \quad (4.2.48)$$

where we used normalization condition given in equation (4.2.16). Generally the normalization depends on the boundary conditions put on the system. In this case we assumed infinite space boundary conditions, hence the factor of $\sqrt{(2\pi)^3}$. When considering the real basis functions the volume of the system is a typical normalization factor. The degeneracy on \mathbf{k} introduces a similar degeneracy on the creation and annihilation operators, on this footing we may write the slowly evolving quantized displaced electric field as:

$$\tilde{\mathbf{D}}^-(\mathbf{r}, t) = i \sum_{\mathbf{k}s} \sqrt{\frac{\hbar\omega_{\mathbf{k}}\epsilon(\mathbf{r})\epsilon_0}{2(2\pi)^3}} \hat{a}_{\mathbf{k}} e^{-i(\omega_{\mathbf{k}} - \omega_L)t} \mathbf{e}_{\mathbf{k}s}(\mathbf{r}) e^{i\mathbf{k}(\mathbf{r}) \cdot \mathbf{r}} \quad (4.2.49a)$$

$$\tilde{\mathbf{D}}^+(\mathbf{r}, t) = -i \sum_{\mathbf{k}} \sqrt{\frac{\hbar\omega_{\mathbf{k}}\epsilon(\mathbf{r})\epsilon_0}{2(2\pi)^3}} \hat{a}_{\mathbf{k}}^\dagger e^{i(\omega_{\mathbf{k}} - \omega_L)t} \mathbf{e}_{\mathbf{k}s}(\mathbf{r}) e^{-i\mathbf{k}(\mathbf{r}) \cdot \mathbf{r}} \quad (4.2.49b)$$

That we only consider the slowly varying field, will become apparent later. Let us then calculate the equal space commutation relation regarding the displaced electric field.

$$\begin{aligned}
[\tilde{D}_r^-(\mathbf{r}, t); \tilde{D}_s^+(\mathbf{r}, t')] &= \sum_{\substack{\mathbf{k}\mathbf{k}' \\ s s'}} \frac{\hbar\epsilon(\mathbf{r})\epsilon_0}{2(2\pi)^3} \sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}} (\mathbf{e}_{\mathbf{k}s}(\mathbf{r}) \cdot \mathbf{e}_r)(\mathbf{e}_{\mathbf{k}'s'}(\mathbf{r}) \cdot \mathbf{e}_s) e^{-i((\omega_{\mathbf{k}}-\omega_L)t - (\omega_{\mathbf{k}'}-\omega_L)t')} [\hat{a}_{\mathbf{k}s}; \hat{a}_{\mathbf{k}'s'}^\dagger] \\
&= \sum_{\substack{\mathbf{k}\mathbf{k}' \\ s s'}} \frac{\hbar\epsilon(\mathbf{r})\epsilon_0}{2(2\pi)^3} \sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}} (\mathbf{e}_{\mathbf{k}s}(\mathbf{r}) \cdot \mathbf{e}_r)(\mathbf{e}_{\mathbf{k}'s'}(\mathbf{r}) \cdot \mathbf{e}_s) e^{-i((\omega_{\mathbf{k}}-\omega_L)t - (\omega_{\mathbf{k}'}-\omega_L)t')} \delta_{\mathbf{k}\mathbf{k}'} \delta_{ss'} \\
&= \sum_{\mathbf{k}s} \frac{\hbar\omega_{\mathbf{k}}\epsilon(\mathbf{r})\epsilon_0}{2(2\pi)^3} (\mathbf{e}_{\mathbf{k}s}(\mathbf{r}) \cdot \mathbf{e}_r)(\mathbf{e}_{\mathbf{k}s}(\mathbf{r}) \cdot \mathbf{e}_s) e^{-i(\omega_{\mathbf{k}}-\omega_L)(t-t')} \tag{4.2.50a}
\end{aligned}$$

In order to continue, we chose a coordinate system such that the unit vector $\mathbf{e}_r = \mathbf{e}_z$ and if $r \neq s$ such that $\mathbf{e}_s = \mathbf{e}_x$. We may also chose our unit vector $\mathbf{e}_{\mathbf{k}1}(\mathbf{r})$ such that it lies in a plane spanned by the vector $\mathbf{k}(\mathbf{r})$ and \mathbf{e}_z . Writing the unit vector $\mathbf{e}_{\mathbf{k}1}(\mathbf{r})$ in polar coordinates we may calculate:

$$\mathbf{e}_{\mathbf{k}1}(\mathbf{r}) = \begin{pmatrix} \sin \theta \sin \phi \\ \sin \theta \cos \phi \\ \cos \theta \end{pmatrix}$$

and

$$\mathbf{e}_{\mathbf{k}2}(\mathbf{r}) = \mathbf{e}_{\mathbf{k}1}(\mathbf{r}) \times \mathbf{e}_z = \begin{pmatrix} \sin \theta \sin \phi \\ \sin \theta \cos \phi \\ \cos \theta \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi \\ -\sin \theta \sin \phi \\ 0 \end{pmatrix}.$$

This means that when integrating over all of space in polar coordinates the contribution from $\mathbf{e}_{\mathbf{k}2}(\mathbf{r})$ vanish, since $\mathbf{e}_{\mathbf{k}2} \cdot \mathbf{e}_z = 0$ and the equation (4.2.50a) reduce to.

$$[\tilde{D}_r^-(\mathbf{r}, t); \tilde{D}_s^+(\mathbf{r}, t')] = \sum_{\mathbf{k}} \frac{\hbar\omega_{\mathbf{k}}\epsilon(\mathbf{r})\epsilon_0}{2(2\pi)^3} (\mathbf{e}_{\mathbf{k}1}(\mathbf{r}) \cdot \mathbf{e}_r)(\mathbf{e}_{\mathbf{k}1}(\mathbf{r}) \cdot \mathbf{e}_s) e^{-i(\omega_{\mathbf{k}}-\omega_L)(t-t')}$$

If $r \neq s$ we may similarly argue that the integral will vanish since $\mathbf{e}_{\mathbf{k}1}(\mathbf{r}) \cdot \mathbf{e}_x = \sin \theta \sin \phi$ vanish when integrated over ϕ .

$$\int_0^{2\pi} d\phi \sin \phi = \int_0^{2\pi} d\phi \cos \phi = 0$$

Writing the sum as an integral, we have:

$$[\tilde{D}_r^-(\mathbf{r}, t); \tilde{D}_s^+(\mathbf{r}, t')] = \delta_{rs} \int dk^3 \frac{\hbar \omega_{\mathbf{k}} \epsilon(\mathbf{r}) \epsilon_0}{2(2\pi)^3} (\mathbf{e}_{\mathbf{k}1}(\mathbf{r}) \cdot \mathbf{e}_z)^2 e^{-i(\omega_{\mathbf{k}} - \omega_L)(t-t')}.$$

This integral we may readily solve, by changing to polar coordinates, and find.

$$[\tilde{D}_r^-(\mathbf{r}, t); \tilde{D}_s^+(\mathbf{r}, t')] = \delta_{rs} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} d\phi d\theta dk k^2 \sin \theta \frac{\hbar \omega_{\mathbf{k}} \epsilon(\mathbf{r}) \epsilon_0}{2(2\pi)^3} \cos^2 \theta e^{-i(\omega_{\mathbf{k}} - \omega_L)(t-t')}.$$

Integrating the variable ϕ out, and changing $\eta = \cos \theta$ we find,

$$\begin{aligned} [\tilde{D}_r^-(\mathbf{r}, t); \tilde{D}_s^+(\mathbf{r}, t')] &= \delta_{rs} \int_{-1}^1 \int_0^{\infty} d\eta dk \eta^2 k^2 \frac{\hbar \omega_{\mathbf{k}} \epsilon(\mathbf{r}) \epsilon_0}{2(2\pi)^2} e^{-i(\omega_{\mathbf{k}} - \omega_L)(t-t')} \\ &= \delta_{rs} \int_0^{\infty} dk k^2 \frac{\hbar \omega_{\mathbf{k}} \epsilon(\mathbf{r}) \epsilon_0}{12\pi^2} e^{-i(\omega_{\mathbf{k}} - \omega_L)(t-t')} \end{aligned}$$

We then assume that the light in consideration is propagating with a well defined carrier frequency, and we therefore approximate the mode frequency $\omega_{\mathbf{k}} \approx \omega_L$, when not appearing in a phase. Using relation (4.2.47), and substituting for k we find:

$$[\tilde{D}_r^-(\mathbf{r}, t); \tilde{D}_s^+(\mathbf{r}, t')] = \delta_{rs} \frac{\hbar \omega_L^3 \epsilon(\mathbf{r})^2 \sqrt{\epsilon(\mathbf{r})} \epsilon_0}{c^3 12\pi^2} \int_0^{\infty} d\omega_{\mathbf{k}} e^{-i(\omega_{\mathbf{k}} - \omega_L)(t-t')}.$$

We may, since dealing with a narrow bandwidth light field, expand the integration limits from $-\infty$ to ∞ , thus getting,

$$[\tilde{D}_r^-(\mathbf{r}, t); \tilde{D}_s^+(\mathbf{r}, t')] = \delta_{rs} \frac{\hbar \omega_L^3 \epsilon(\mathbf{r})^2 \sqrt{\epsilon(\mathbf{r})} \epsilon_0}{c^3 6\pi} \delta(t - t').$$

Using relation (4.2.47) again we finally arrive at the equal-space commutation relation given by:

$$[\tilde{D}_r^-(\mathbf{r}, t); \tilde{D}_s^+(\mathbf{r}, t')] = \delta_{rs} \frac{\hbar k_L^3 \epsilon(\mathbf{r}) \epsilon_0}{6\pi} \delta(t - t') \quad (4.2.50b)$$

The relevant commutation relations for a polarized system thus reads:

$$[\hat{\mathbf{A}}^+(\mathbf{r}, t); \hat{\mathbf{D}}^+(\mathbf{r}', t)] = [\hat{\mathbf{A}}^-(\mathbf{r}, t); \hat{\mathbf{D}}^-(\mathbf{r}', t)] = 0 \quad (4.2.51a)$$

$$[\hat{\mathbf{A}}^+(\mathbf{r}, t); \hat{\mathbf{D}}^-(\mathbf{r}', t)] = -\frac{i\hbar}{2} \bar{\delta}^T(\mathbf{r}, \mathbf{r}') \quad (4.2.51b)$$

$$[\tilde{D}_r^-(\mathbf{r}, t); \tilde{D}_s^+(\mathbf{r}, t')] = \delta_{rs} \frac{\hbar k_L^3 \epsilon(\mathbf{r}) \epsilon_0}{6\pi} \delta(t - t') \quad (4.2.51c)$$

We now have the needed information settled, regarding an expansion of fields in the, for a dispersive system, appropriate basis functions. We know that we may safely split the field $\hat{\mathbf{D}}(\mathbf{r}, t)$ into a photon creating part and a photon annihilating part, oscillating clockwise and counter clockwise in time: (4.2.40). We know how the commutation relations on these terms behave: (4.2.51b), and we have detailed information regarding the connection between creation and annihilation operators and field: (4.2.39). Let us now return to the real classical Hamiltonian (4.1.3).

4.3 Effective Quantized Hamiltonian

In this section we derive an effective quantized Hamiltonian, capable of describing our system. Let us remind our selves of the Classical Hamiltonian (4.1.3)

$$\mathcal{H} = \int d^3r \left\{ \frac{1}{2} \left(\frac{\mathbf{\Pi}^2}{\epsilon_0} + \frac{(\nabla \times \mathbf{A})^2}{\mu_0} + \frac{e^2 \mathcal{X}^2}{\epsilon_0 m} \right) - \nabla \phi \cdot \mathbf{\Pi} - \frac{e}{\epsilon_0} \mathbf{\Pi} \cdot \boldsymbol{\chi} + \frac{\boldsymbol{\theta}^2}{2m} + eU_{\text{atom}}(\boldsymbol{\chi}) \right\}. \quad (4.3.1)$$

When quantizing we merely impose the commutation relations from equation (4.2.51b) onto the fields. For the physical fields $\hat{\mathbf{A}}(\mathbf{r}, t)$ and $\hat{\mathbf{D}}(\mathbf{r}, t)$ we find:

$$\begin{aligned} [\hat{\mathbf{A}}(\mathbf{r}, t); \hat{\mathbf{D}}(\mathbf{r}, t)] &= [\hat{\mathbf{A}}^+(\mathbf{r}, t) + \hat{\mathbf{A}}^-(\mathbf{r}, t); \hat{\mathbf{D}}^+(\mathbf{r}, t) + \hat{\mathbf{D}}^-(\mathbf{r}, t)] \\ &= [\hat{\mathbf{A}}^+(\mathbf{r}, t); \hat{\mathbf{D}}^-(\mathbf{r}, t)] + [\hat{\mathbf{A}}^-(\mathbf{r}, t); \hat{\mathbf{D}}^+(\mathbf{r}, t)] \\ &= -\frac{i\hbar}{2} \bar{\delta}^T(\mathbf{r}, \mathbf{r}') - \frac{i\hbar}{2} \bar{\delta}^T(\mathbf{r}, \mathbf{r}') \\ &= -i\hbar \bar{\delta}^T(\mathbf{r}, \mathbf{r}') \end{aligned} \quad (4.3.2)$$

This result is equivalent to the canonical commutation relations in equation (4.1.7), though we in this version explicitly require transversality. This can be traced back to the Gauge we chose

$\nabla\phi(\mathbf{r}, t) = 0$. What we in this version acquired is the additional commutation relations, when considering only the photon creating or annihilating part of the field. Assuming the field may be split in a positively and negatively oscillating part devise a direct method of acquiring the corresponding commutation relation. This has been put in Appendix A. We will be working in the Gauge $\nabla\phi(\mathbf{r}, t) = 0$ and neglect the term $\frac{e^2\chi^2}{\epsilon_0 m}$ following the discussion at page 32. Identifying the displaced electric field, the quantized Hamiltonian reads

$$\begin{aligned}\hat{\mathcal{H}} &= \hat{\mathcal{H}}_{\text{Field}} + \hat{\mathcal{H}}_{\text{Int}} + \hat{\mathcal{H}}_{\text{Atoms}} \\ &= \int d^3r \left\{ \frac{1}{2} \left(\frac{\hat{\mathbf{D}}^2}{\epsilon_0} + \frac{(\nabla \times \hat{\mathbf{A}})^2}{\mu_0} \right) + \frac{e}{\epsilon_0} \hat{\mathbf{D}} \cdot \chi + \frac{\theta^2}{2m} + eU_{\text{atom}}(\chi) \right\}.\end{aligned}\quad (4.3.3)$$

We suppressed space and time dependence to shorten the notation.

Let us consider the part of the Hamiltonian describing the atoms, and expand this in eigenfunctions for the Hamiltonian.

$$\hat{\mathcal{H}}_{\text{atoms}} = \int d^3r \frac{\hat{\theta}^2}{2m} + eU_{\text{atom}}(\hat{\chi}) = \sum_j^{\text{Atoms}} \sum_n E_n^j \langle n|_j \quad (4.3.4)$$

The physical setup is the same as in the one dimensional case, hence the atoms all submit to the level scheme depicted in Figure 2.1. We therefore copy the results derived in the section regarding the atoms, and find:

$$\hat{\mathcal{H}}_{\text{atoms}} = \sum_j^{\text{Atoms}} \frac{1}{2} \hbar \omega_0 (\hat{\sigma}_{33}^j - \hat{\sigma}_{11}^j + \hat{\sigma}_{44}^j - \hat{\sigma}_{22}^j) \quad (4.3.5)$$

Considering the interaction Hamiltonian

$$\hat{\mathcal{H}}_{\text{Int}} = - \int d^3r \frac{e}{\epsilon_0} \hat{\mathbf{D}} \cdot \chi, \quad (4.3.6)$$

we identify the operator $e\chi$ with the polarization operator $\hat{\mathbf{P}}(\mathbf{r}, t)$ defined in equation (2.2.3).

$$\hat{\mathcal{H}}_{\text{Int}} = - \int d^3r \frac{1}{\epsilon_0} \hat{\mathbf{D}}(\mathbf{r}, t) \cdot \hat{\mathbf{P}}(\mathbf{r}, t) \quad (4.3.7)$$

Having established the necessary information regarding the field $\hat{\mathbf{D}}(\mathbf{r}, t)$, we thereby mean the decomposition in a slowly oscillating- and fast oscillating part, we may perform the adiabatic elimination as done in section 2.2.1. Keeping in mind the factor of ϵ_0 as well as the fact that we are not dealing with electric fields but with electric displacement, we may write the effective

interaction Hamiltonian as:

$$\hat{\mathcal{H}}_{\text{Eff}}^{\text{Int}} = \sum_j^{\text{atoms}} \frac{-2\beta}{\epsilon_0} \hat{\mathbf{D}}^+(\mathbf{r}_j, t) \bar{\alpha}_j \hat{\mathbf{D}}^-(\mathbf{r}_j, t) \quad (4.3.8)$$

where

$$\beta \equiv \frac{g^2 d^2}{\epsilon_0 \hbar \Delta} \quad \bar{\alpha}_j \equiv \frac{1}{2} \bar{I}^j - i \hat{\mathbf{J}}^j \times \quad (4.3.9)$$

The operator \bar{I}^j is the identity operator concerning the j -th atom and $\hat{\mathbf{J}}^j$ is an operator measuring spin of the j -th atom. The operator is defined in equation (2.2.19), (2.2.20) and (2.2.21). In the adiabatic elimination scheme we only considered motion happening at the ground level, this effectively leaves out $\hat{\mathcal{H}}_{\text{Atom}}$ from considerations. We conclude this chapter by writing out the *Effective Hamiltonian* describing the evolution of our system.

$$\hat{\mathcal{H}}_{\text{Eff}} = \frac{1}{2} \int d^3 r \left\{ \frac{\hat{\mathbf{D}}(\mathbf{r}, t)^2}{\epsilon_0} + \frac{(\nabla \times \hat{\mathbf{A}}(\mathbf{r}, t))^2}{\mu_0} \right\} - \sum_j^{\text{atoms}} \frac{2\beta}{\epsilon_0} \hat{\mathbf{D}}^+(\mathbf{r}_j, t) \bar{\alpha}_j \hat{\mathbf{D}}^-(\mathbf{r}_j, t) \quad (4.3.10)$$

Equation of motion for the system

Having the quantized effective Hamiltonian at hand, we proceed by using *the Heisenberg equation of motion* to find the relevant differential equations describing the evolution of our system. Having the set of differential equations we will devise methods to solve them. Imagine some differential operator D involved in a differential equation we want to solve.

$$D\psi = \rho$$

Inspired by algebra, we could just work with the inverse differential operator D^{-1} on both sides of the equation, and the problem would be solved.

$$\psi = D^{-1}\rho$$

In frames of analysis this of course complicates a little, but we will show that there exist meaningful ways to define relevant inverse differential operators, generally known as *Green's functions*. The method of solving the differential equation, as we will see, correspond to a transformation of the equations from a differential form to an integral form. These integral equations can then be approximately solved using *Born approximations*. This implies that the equations we consider, must be suitable for a perturbative treatment. We will therefore strive to find a differential equation describing our system on the form,

$$D\psi = \beta\rho \tag{5.0.1}$$

where β is a small quantity.

5.1 Equations of motion for the field $\hat{\mathbf{D}}(\mathbf{r}, t)$

Physically we are dealing with a dispersive system, where the phase of the light changes as it propagates through an atomic gas. We expect to find a wave equation similar to equation (4.2.8). We also expect to find terms describing the possible scattering processes which we will treat as perturbations. Let us first consider the quantized electric displacement $\hat{\mathbf{D}}(\mathbf{r}, t)$. Considering the field component wise, and using *the Heisenberg equation of motion*, we find the first Maxwell equation.

$$\frac{\partial}{\partial t} \hat{\mathbf{D}}(\mathbf{r}, t) = \frac{1}{\mu_0} \nabla \times \hat{\mathbf{B}}(\mathbf{r}, t). \quad (5.1.1)$$

We used the definition of the magnetic field, given in equation (3.1.5b). The detailed calculations has been put in Appendix C.6. The Maxwell equation is also found for the general polarized media, equation (4.2.7a). Similarly we may calculate the corresponding equation regarding the magnetic field.

$$\frac{\partial}{\partial t} \hat{\mathbf{B}}(\mathbf{r}, t) = -\frac{1}{\epsilon_0} \nabla \times \hat{\mathbf{D}}(\mathbf{r}, t) + \sum_j^{\text{atoms}} \frac{\beta}{\epsilon_0} \nabla \times \bar{\delta}^{\text{T}}(\mathbf{r}, \mathbf{r}_j) \cdot \left\{ \frac{1}{2} \hat{\mathbf{D}}(\mathbf{r}_j, t) + i \hat{\mathbf{J}}^j \times (\hat{\mathbf{D}}^+(\mathbf{r}_j, t) - \hat{\mathbf{D}}^-(\mathbf{r}_j, t)) \right\} \quad (5.1.2)$$

We have put in the result of the calculation only, and again identified the magnetic field $\hat{\mathbf{B}}(\mathbf{r}, t)$. The calculation, instructive as it may be, has been put in Appendix C.6.2 We combine the two equations (5.1.1) and (5.1.2), and arrive at the following wave equation.

$$\left(\frac{\partial^2}{\partial t^2} + c^2 \nabla \times \nabla \times \right) \hat{\mathbf{D}}(\mathbf{r}, t) = \sum_j^{\text{atoms}} c^2 \beta \nabla \times \nabla \times \bar{\delta}^{\text{T}}(\mathbf{r}, \mathbf{r}_j) \cdot \left\{ \frac{1}{2} \hat{\mathbf{D}}(\mathbf{r}_j, t) + i \hat{\mathbf{J}}^j \times (\hat{\mathbf{D}}^+(\mathbf{r}_j, t) - \hat{\mathbf{D}}^-(\mathbf{r}_j, t)) \right\} \quad (5.1.3)$$

In principle this equation tells us everything there is to know about the evolution of the electric displacement $\hat{\mathbf{D}}(\mathbf{r}, t)$. However our lacking abilities to solve this kind of equations exactly by means of first principles, requires a different line of approach. We already argued that we wanted to treat the source, that is the right hand side of equation (5.1.3) as a perturbation. Unfortunately the equation (5.1.3) in its current form, is not suited for perturbative treatment. The physical situation described by the source term is the effect of sending light through a media. We know this effect is not small in nature, the light acquires a considerable phase shift. In the following we will devise a method to extract the dispersive part of equation (5.1.3) leaving the remaining right hand side as a perturbation.

Let us define a field describing the density of atoms in the gas, and a field describing the spin of the atoms:

$$\rho(\mathbf{r}) = \sum_j^{\text{atoms}} \delta(\mathbf{r} - \mathbf{r}_j), \quad (5.1.4)$$

$$\hat{\mathbf{J}}(\mathbf{r}) = \sum_j^{\text{atoms}} \delta(\mathbf{r} - \mathbf{r}_j) \hat{\mathbf{J}}^j. \quad (5.1.5)$$

Using these definitions, the equation (5.1.3) may be written as:

$$\begin{aligned} & \left(\frac{\partial^2}{\partial t^2} + c^2 \nabla \times \nabla \times \right) \hat{\mathbf{D}}(\mathbf{r}, t) \\ &= \int d^3 r' c^2 \beta \nabla \times \nabla \times \bar{\delta}^T(\mathbf{r}, \mathbf{r}') \cdot \left\{ \frac{1}{2} \rho(\mathbf{r}') \hat{\mathbf{D}}(\mathbf{r}', t) + i \hat{\mathbf{J}}(\mathbf{r}') \times (\hat{\mathbf{D}}^+(\mathbf{r}', t) - \hat{\mathbf{D}}^-(\mathbf{r}', t)) \right\} \end{aligned} \quad (5.1.6)$$

Further defining the mean density $\langle \rho(\mathbf{r}) \rangle_{\text{Mean}}$, and the deviation from this:

$$\lambda(\mathbf{r}) = \rho(\mathbf{r}) - \langle \rho(\mathbf{r}) \rangle_{\text{Mean}}, \quad (5.1.7)$$

equation (5.1.6) may be written as:

$$\begin{aligned} & \left(\frac{\partial^2}{\partial t^2} + c^2 \nabla \times \nabla \times \right) \hat{\mathbf{D}}(\mathbf{r}, t) \\ &= \int d^3 r' c^2 \beta \nabla \times \nabla \times \bar{\delta}^T(\mathbf{r}, \mathbf{r}') \cdot \left\{ \frac{1}{2} (\rho(\mathbf{r}') - \langle \rho(\mathbf{r}') \rangle_{\text{Mean}} + \langle \rho(\mathbf{r}') \rangle_{\text{Mean}}) \hat{\mathbf{D}}(\mathbf{r}', t) \right. \\ & \quad \left. + i \hat{\mathbf{J}}(\mathbf{r}') \times (\hat{\mathbf{D}}^+(\mathbf{r}', t) - \hat{\mathbf{D}}^-(\mathbf{r}', t)) \right\} \\ &= \int d^3 r' c^2 \beta \nabla \times \nabla \times \bar{\delta}^T(\mathbf{r}, \mathbf{r}') \cdot \left\{ \frac{1}{2} \lambda(\mathbf{r}') \hat{\mathbf{D}}(\mathbf{r}', t) + i \hat{\mathbf{J}}(\mathbf{r}') \times (\hat{\mathbf{D}}^+(\mathbf{r}', t) - \hat{\mathbf{D}}^-(\mathbf{r}', t)) \right\} \\ & \quad + \int d^3 r' c^2 \frac{\beta}{2} \nabla \times \nabla \times \bar{\delta}^T(\mathbf{r}, \mathbf{r}') \cdot \langle \rho(\mathbf{r}') \rangle_{\text{Mean}} \hat{\mathbf{D}}(\mathbf{r}', t). \end{aligned} \quad (5.1.8)$$

Assuming that the mean density varies slowly with position, we may set the product, $\langle \rho(\mathbf{r}') \rangle_{\text{Mean}} \tilde{\mathbf{D}}^+(\mathbf{r}', t)$, to be transverse. By slowly we mean compared to the wavelength of the light, generally we would like the direction of propagation of the light to be perpendicular to the direction of the field vector, \mathbf{D} . Using properties regarding the transverse delta function, we find the last term to reduce to:

$$c^2 \frac{\beta}{2} \nabla \times \nabla \times \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \hat{\mathbf{D}}(\mathbf{r}, t). \quad (5.1.9)$$

Defining $\chi(\mathbf{r}) \equiv \frac{\beta}{2} \langle \rho(\mathbf{r}) \rangle_{\text{Mean}}$ we may write:

$$\begin{aligned} & \left(\frac{\partial^2}{\partial t^2} + c^2 \nabla \times \nabla \times \right) \hat{\mathbf{D}}(\mathbf{r}, t) \\ &= \int d^3 r' c^2 \beta \nabla \times \nabla \times \bar{\delta}^T(\mathbf{r}, \mathbf{r}') \cdot \left\{ \frac{1}{2} \lambda(\mathbf{r}') \hat{\mathbf{D}}(\mathbf{r}', t) + i \hat{\mathbf{J}}(\mathbf{r}') \times (\hat{\mathbf{D}}^+(\mathbf{r}', t) - \hat{\mathbf{D}}^-(\mathbf{r}', t)) \right\} \\ & \quad + c^2 \nabla \times \nabla \times \chi(\mathbf{r}) \hat{\mathbf{D}}(\mathbf{r}', t). \end{aligned} \quad (5.1.10)$$

Defining $\epsilon^{-1}(\mathbf{r}) \equiv 1 - \chi(\mathbf{r})$, we find

$$\begin{aligned} & \left(\frac{\partial^2}{\partial t^2} + c^2 \nabla \times \nabla \times \epsilon^{-1}(\mathbf{r}) \right) \hat{\mathbf{D}}(\mathbf{r}, t) \\ &= \int d^3 r' c^2 \beta \nabla \times \nabla \times \bar{\delta}^T(\mathbf{r}, \mathbf{r}') \cdot \left\{ \frac{1}{2} \lambda(\mathbf{r}') \hat{\mathbf{D}}(\mathbf{r}', t) + i \hat{\mathbf{J}}(\mathbf{r}') \times (\hat{\mathbf{D}}^+(\mathbf{r}', t) - \hat{\mathbf{D}}^-(\mathbf{r}', t)) \right\}, \end{aligned} \quad (5.1.11)$$

where we used the approximation made in (5.1.9). In this respect we have extracted the main effect of the media from the source term, i.e. the right hand side of equation (5.1.3). The right hand side of equation (5.1.11) we will now treat as a perturbation. Let us compare this equation with the equation of motion derived for a polarized media (4.2.8). The harmonic part of equation (5.1.11) thus describes the evolution of a field in a polarized media, described in section 4.2, thus the harmonic solution of the field expanded in terms of creation- and annihilation operators are given in equation (4.2.39). We also see that the general factor $\epsilon(\mathbf{r})$ used in section 4.2 is in our system given by:

$$\epsilon(\mathbf{r}) = \frac{1}{1 - \chi(\mathbf{r})} = \frac{1}{1 - \frac{\beta}{2} \langle \rho(\mathbf{r}) \rangle_{\text{Mean}}} \quad (5.1.12)$$

The equation (5.1.11) constitutes one half of the governing set of differential equations describing the interaction between light and matter in our approximation scheme. The second important equation is the differential equation describing the evolution of the spin of the atoms.

5.2 Equation of motion for the spin of the atoms

As with the displaced electric field, we may also derive an equation of motion for the spin of the atoms, using *Heisenberg equation of motion*. We find the governing equation regarding the spin

of the k -th atom to be the following:

$$\frac{d}{dt} \hat{\mathbf{J}}^k = \frac{2i\beta}{\hbar\epsilon_0} \hat{\mathbf{J}}^k \times \hat{\mathbf{D}}^+(\mathbf{r}_k, t) \times \hat{\mathbf{D}}^-(\mathbf{r}_k, t). \quad (5.2.1)$$

Detailed calculations can be found in Appendix C.6.3.

In the following section we will pay particular attention to the wave equation (5.1.3). We will derive transformation from the differential form to integral form, using Greens function. We will consider the problem in terms of the slowly oscillating field $\tilde{\mathbf{D}}^+(\mathbf{r}, t)$ and $\tilde{\mathbf{D}}^-(\mathbf{r}, t)$, this allow us to approximate the second order time derivative wave equation, with a first order one. The reason for this change is most easily described by considering the simple second order time differential equation.

$$\frac{d^2}{dt^2} f(t) = a \quad (5.2.2)$$

We know the solution to this equation is generally given as:

$$f(t) = \frac{1}{2}at^2 + \left. \frac{df(t)}{dt} \right|_{t_0} t + f(t_0) \quad (5.2.3)$$

This illustrates the problem of a second order differential equation, not only will we have to specify the initial condition for the function $f(t)$, we also need to know the initial conditions regarding the time derivative of $f(t)$. Even though the differential equation we are solving is a lot more complex than the simple one (5.2.2), we find by consulting the book (46; 47), that the three dimensional wave-equation does indeed require knowledge of the first order time derivative of the field $\hat{\mathbf{D}}(\mathbf{r}, t)$. In this respect by changing the wave-equation (5.1.11) to a first order differential equation in time, we hope the general solution simplify.

5.3 Solution in terms of Greens function

Consider a general waveequation with some source $\rho(\mathbf{r}, t)$

$$\left(\frac{\partial^2}{\partial t^2} + c^2 \nabla \times \nabla \times \right) \psi(\mathbf{r}, t) = \rho(\mathbf{r}, t) \quad (5.3.1)$$

Suppose we may split the field $\psi(\mathbf{r}, t)$ and the source $\rho(\mathbf{r}, t)$ in a slow oscillating part and a fast oscillating part,

$$\psi(\mathbf{r}, t) = \tilde{\psi}^+(\mathbf{r}, t)e^{i\omega_L t} + \tilde{\psi}^-(\mathbf{r}, t)e^{-i\omega_L t} \quad (5.3.2a)$$

$$\rho(\mathbf{r}, t) = \tilde{\rho}^+(\mathbf{r}, t)e^{i\omega_L t} + \tilde{\rho}^-(\mathbf{r}, t)e^{-i\omega_L t} \quad (5.3.2b)$$

The time differentiation may then be written as:

$$\begin{aligned} \frac{\partial^2}{\partial t^2}(\tilde{\psi}^+(\mathbf{r}, t)e^{i\omega_L t}) &= \frac{\partial}{\partial t}\left(\frac{\partial \tilde{\psi}^+(\mathbf{r}, t)}{\partial t}e^{i\omega_L t} + i\omega_L \tilde{\psi}^+(\mathbf{r}, t)e^{i\omega_L t}\right) \\ &= \frac{\partial^2 \tilde{\psi}^+(\mathbf{r}, t)}{\partial t^2}e^{i\omega_L t} + 2i\omega_L \frac{\partial \tilde{\psi}^+(\mathbf{r}, t)}{\partial t}e^{i\omega_L t} - \omega_L^2 \tilde{\psi}^+(\mathbf{r}, t)e^{i\omega_L t} \\ &= e^{i\omega_L t}\left(\frac{\partial}{\partial t} + i\omega_L\right)^2 \tilde{\psi}^+(\mathbf{r}, t). \end{aligned} \quad (5.3.3)$$

A similar equation is found for the negatively oscillating term. Using this we may rewrite equation (5.3.1) to two equations each concerning a component of the field.

$$\left(\left(\frac{\partial}{\partial t} + i\omega_L\right)^2 + c^2 \nabla \times \nabla \times\right) \tilde{\psi}^+(\mathbf{r}, t) = \tilde{\rho}^+(\mathbf{r}, t) \quad (5.3.4a)$$

$$\left(\left(\frac{\partial}{\partial t} - i\omega_L\right)^2 + c^2 \nabla \times \nabla \times\right) \tilde{\psi}^-(\mathbf{r}, t) = \tilde{\rho}^-(\mathbf{r}, t) \quad (5.3.4b)$$

where we divided out the phase $e^{\pm i\omega_L t}$. From the assumption that the fields $\tilde{\psi}^+(\mathbf{r}, t)$ and $\tilde{\psi}^-(\mathbf{r}, t)$ evolve slow, we set,

$$\frac{\partial^2 \tilde{\psi}^+(\mathbf{r}, t)}{\partial t^2} = \frac{\partial^2 \tilde{\psi}^-(\mathbf{r}, t)}{\partial t^2} = 0. \quad (5.3.5)$$

Using this, the equations (5.3.4) approximate to the following:

$$\left(+2i\omega_L \frac{\partial}{\partial t} - \omega_L^2 + c^2 \nabla \times \nabla \times\right) \tilde{\psi}^+(\mathbf{r}, t) = \tilde{\rho}^+(\mathbf{r}, t) \quad (5.3.6a)$$

$$\left(-2i\omega_L \frac{\partial}{\partial t} - \omega_L^2 + c^2 \nabla \times \nabla \times\right) \tilde{\psi}^-(\mathbf{r}, t) = \tilde{\rho}^-(\mathbf{r}, t) \quad (5.3.6b)$$

In the following we will consider the integral solution to the type of equations (5.3.6), also known as diffusion equations. We have already solved a simpler differential equation (2.2.17), the following description is merely a generalization of this type of solution. In equation (2.2.16) we worked with the inverse of the differential operator $\frac{d}{dt}$ on both sides of the equation. Similarly we will find inverse operators to the much more complex differential operators given in equation (5.3.6). These inverse operators are called Green's functions. In the subsequent section we describe these

Green's functions, and the solution they provide. The section follows ideas presented in, e.g., (46; 47). We will show that for an three dimensional diffusion equation, the general solution is not determined by the time derivative of the initial field, only by the initial field itself. We will do this derivation in some detail since the calculation is essential for the solution of the problem at hand, and also since the calculation does not appear in standard text books.*

5.3.1 Properties for the Green's function

To solve the partial differential equation (5.3.6), we need a proper Green's function. Since we are dealing with transverse fields, we will use the identity:

$$\nabla \times \nabla \times = -\nabla^2 \quad (5.3.7)$$

The Green's function is in fact a tensor[†], and has to obey the following differential equation. We will consider the positively oscillating component first.

$$\left(2i\omega_L \frac{\partial}{\partial t} - \omega_L^2 - c^2 \nabla^2\right) \bar{G}^+(\mathbf{r}, t | \mathbf{r}_0, t_0) = \bar{I} \delta(\mathbf{r} - \mathbf{r}_0) \delta(t - t_0) \quad (5.3.8)$$

A similar equation applies to the negative frequency component. First we consider the general structure of a Green's function, and describe a possible way of deriving the desired Green's function. When deriving the general Green's function, it is a good idea to consider the following eigenvalue equation:

$$\left(2i\omega_L \frac{\partial}{\partial t} - \omega_L^2 - c^2 \nabla^2\right) \mathbf{F}_n(\mathbf{r}, t) = \omega_n^2 \mathbf{F}_n(\mathbf{r}, t)$$

or,

$$\left(2i\omega_L \frac{\partial}{\partial t} - c^2 \nabla^2\right) \mathbf{F}_n(\mathbf{r}, t) = (\omega_L^2 + \omega_n^2) \mathbf{F}_n(\mathbf{r}, t). \quad (5.3.9)$$

Let us now define the differential operator D :

$$D \equiv 2i\omega_L \frac{\partial}{\partial t} - c^2 \nabla^2 \quad (5.3.10)$$

*The author has found this derivation nowhere, it is standard to consider the waveequation, apparently not the three dimensional diffusion equation.

[†]A physical tensor transforming under the principle of general covariance, see, e.g., (48)

To ensure that the set $\{\mathbf{F}_n(\mathbf{r}, t)\}$ of eigenfunctions to D may be considered complete, we need to check if the differential operator L is hermitian. We define the inner product on the space of transverse fields in the usual way, time included:

$$\langle \psi(\mathbf{r}, t) | \phi(\mathbf{r}, t) \rangle = \iint d^3r dt \psi(\mathbf{r}, t) \cdot \phi^*(\mathbf{r}, t). \quad (5.3.11)$$

Let us now check for the adjoint of the operator D . To shorten notation we suppress time and space dependence on the fields.

$$\begin{aligned} \langle \phi | D\psi \rangle &= \iint d^3r dt \phi \cdot (L\psi)^* = \iint d^3r dt \phi_n \left((-2i\omega_L \frac{\partial}{\partial t} - c^2 \nabla^2) \psi^* \right)_n \\ &= \iint d^3r dt \phi_n \left(-2i\omega_L \frac{\partial}{\partial t} - c^2 \nabla^2 \right) \psi_n^*. \end{aligned}$$

Using partial integration once on time, and twice on space, assuming usual boundary conditions on the fields (vanish at infinity), we get:

$$\begin{aligned} \langle \phi | D\psi \rangle &= \iint d^3r dt \left(2i\omega_L \frac{\partial}{\partial t} - c^2 \nabla^2 \right) \phi_n \psi_n^* \\ &= \iint d^3r dt (L\phi) \cdot \psi^* \end{aligned} \quad (5.3.12)$$

$$= \langle D\phi | \psi \rangle. \quad (5.3.13)$$

We thus see that the differential operator L is hermitian, and therefore the set of eigenfunctions to the operator span the functional space in consideration.

We will now construct the general Green's function as given in terms of these eigenfunctions. Suppose the Green's function can be written as a juxtaposition[‡] of eigenfunctions for the operator L concerning the variables \mathbf{r}, t , that is $\{\mathbf{F}_n(\mathbf{r}, t)\}$ and some functions on the variables \mathbf{r}_0, t_0 , $\{\mathbf{c}_n(\mathbf{r}_0, t_0)\}$. We write:

$$\bar{\bar{G}}^+(\mathbf{r}, t | \mathbf{r}_0, t_0) = \sum_n \mathbf{F}_n(\mathbf{r}, t) \mathbf{c}_n(\mathbf{r}_0, t_0) \quad (5.3.14)$$

We then calculate:

$$\begin{aligned} L\bar{\bar{G}}^+(\mathbf{r}, t | \mathbf{r}_0, t_0) &= \sum_n L\mathbf{F}_n(\mathbf{r}, t) \mathbf{c}_n(\mathbf{r}_0, t_0) \\ &= \sum_n (\omega_L^2 + \omega_n^2) \mathbf{F}_n(\mathbf{r}, t) \mathbf{c}_n(\mathbf{r}_0, t_0). \end{aligned} \quad (5.3.15a)$$

[‡]the canonical way of making two vectors a tensor. ($V_s W_r = T_{sr}$)

However from definition (5.3.8), we also know:

$$L\bar{\bar{G}}^+(\mathbf{r}, t|\mathbf{r}_0, t_0) = \delta(\mathbf{r} - \mathbf{r}_0)\delta(t - t_0)\bar{\bar{I}} + \omega_L^2\bar{\bar{G}}^+(\mathbf{r}, t|\mathbf{r}_0, t_0). \quad (5.3.15b)$$

Subtracting the two equations (5.3.15a) and (5.3.15b) leaves us with:

$$\begin{aligned} \sum_n \mathbf{c}_n(\mathbf{r}_0, t_0)\omega_n^2\mathbf{F}_n(\mathbf{r}, t) &= \delta(\mathbf{r} - \mathbf{r}_0)\delta(t - t_0)\bar{\bar{I}} \\ &= \sum_n \mathbf{F}_n(\mathbf{r}_0, t_0)\mathbf{F}_n^*(\mathbf{r}, t) \end{aligned} \quad (5.3.15c)$$

We thus see that the Green's function may be represented as:

$$\bar{\bar{G}}^+(\mathbf{r}, t|\mathbf{r}_0, t_0) = \sum_n \frac{1}{\omega_n^2}\mathbf{F}_n(\mathbf{r}_0, t_0)\mathbf{F}_n(\mathbf{r}, t) \quad (5.3.16)$$

A general trick when dealing with Green's functions, is to define the problem in terms of propagators. The reason for this change will be apparent later in the calculations. The redefinition basically means to introduce a cut-off to the Green's function, and only let it describe evolution happening forward in time, hence the name propagator. We therefore restrict:

$$\bar{\bar{G}}^+(\mathbf{r}, t|\mathbf{r}_0, t_0) = 0 \quad \text{for} \quad t < t_0. \quad (5.3.17)$$

The price we pay from this restriction is the apparent loss of ability to say what happened backward in time. In the following calculation however, we show how we may derive the equations telling how the evolution backwards in time and space is going to happen, that is to see how $\bar{\bar{G}}^+$ acts when interchange of prime and unprimed coordinates.

Consider the defining equation for $\bar{\bar{G}}^+$:

$$\left(2i\omega_L\frac{\partial}{\partial t} - c^2\nabla^2 - \omega_L^2\right)\bar{\bar{G}}^+(\mathbf{r}, t|\mathbf{r}_0, t_0) = \delta(\mathbf{r} - \mathbf{r}_0)\delta(t - t_0)\bar{\bar{I}} \quad (5.3.18a)$$

When substituting $t \rightarrow -t$ and $t_0 \rightarrow -t_1$ we find:

$$\left(-2i\omega_L\frac{\partial}{\partial t} - c^2\nabla^2 - \omega_L^2\right)\bar{\bar{G}}^+(\mathbf{r}, -t|\mathbf{r}_1, -t_1) = \delta(\mathbf{r} - \mathbf{r}_1)\delta(t - t_1)\bar{\bar{I}} \quad (5.3.18b)$$

Making the fully contracting inner tensor product equation 5.3.18a with $\bar{\bar{G}}^+(\mathbf{r}, -t|\mathbf{r}_1, -t_1)$ and equa-

tion 5.3.18b with $\bar{G}^+(\mathbf{r}, t|\mathbf{r}_0, t_0)$ and then subtracting gives the following.

$$-c^2\{G_{nm}^+(\mathbf{r}, t|\mathbf{r}_0, t_0)\nabla^2 G_{nm}^+(\mathbf{r}, -t|\mathbf{r}_1, -t_1) - G_{nm}^+(\mathbf{r}, -t|\mathbf{r}_1, -t_1)\nabla^2 G_{nm}^+(\mathbf{r}, t|\mathbf{r}_0, t_0)\} \quad (5.3.19a)$$

$$- \omega_L^2\{G_{nm}^+(\mathbf{r}, t|\mathbf{r}_0, t_0)G_{nm}^+(\mathbf{r}, -t|\mathbf{r}_1, -t_1) - G_{nm}^+(\mathbf{r}, -t|\mathbf{r}_1, -t_1)G_{nm}^+(\mathbf{r}, t|\mathbf{r}_0, t_0)\} \quad (5.3.19b)$$

$$+ 2i\omega_L \frac{\partial}{\partial t}(G_{nm}^+(\mathbf{r}, t|\mathbf{r}_0, t_0)G_{nm}^+(\mathbf{r}, -t|\mathbf{r}_1, -t_1)) \quad (5.3.19c)$$

$$= I_{nm}G_{nm}^+(\mathbf{r}, t|\mathbf{r}_0, t_0)\delta(\mathbf{r} - \mathbf{r}_1)\delta(t - t_1) - I_{nm}G_{nm}^+(\mathbf{r}, -t|\mathbf{r}_1, -t_1)\delta(\mathbf{r} - \mathbf{r}_0)\delta(t - t_0) \quad (5.3.19d)$$

The term (5.3.19b) vanish trivially. We then integrate over time from $t \in] - \infty; t_1^+[$, where $t^+ = t + \epsilon$, for $\epsilon \rightarrow 0$, and integrate over space. The term (5.3.19d) thus gives:

$$I_{nm}G_{nm}(\mathbf{r}_1, t_1|\mathbf{r}_0, t_0) - I_{nm}G_{nm}(\mathbf{r}_0, -t_0|\mathbf{r}_1, -t_1), \quad (5.3.20)$$

and the term (5.3.19c) gives:

$$\begin{aligned} & \iint_{-\infty}^{t_1^+} d^3r dt \ 2i\omega_L \frac{\partial}{\partial t}(G_{nm}^+(\mathbf{r}, t|\mathbf{r}_0, t_0)G_{nm}^+(\mathbf{r}, -t|\mathbf{r}_1, -t_1)) \\ &= \int d^3r \ 2i\omega_L \{G_{nm}^+(\mathbf{r}, t|\mathbf{r}_0, t_0)G_{nm}^+(\mathbf{r}, -t|\mathbf{r}_1, -t_1)\}_{-\infty}^{t_1^+} \\ &= 0. \end{aligned} \quad (5.3.21)$$

This follows from equation (5.3.17), since $G_{nm}^+(\mathbf{r}, -\infty|\mathbf{r}_0, t_0) = 0$ and $G_{nm}^+(\mathbf{r}, -t_1^+|\mathbf{r}_1, -t_1) = 0$. The remaining terms thus reads:

$$\begin{aligned} & \iint_{-\infty}^{t_1^+} d^3r dt \ -c^2\{G_{nm}^+(\mathbf{r}, t|\mathbf{r}_0, t_0)\nabla^2 G_{nm}^+(\mathbf{r}, -t|\mathbf{r}_1, -t_1) - G_{nm}^+(\mathbf{r}, -t|\mathbf{r}_1, -t_1)\nabla^2 G_{nm}^+(\mathbf{r}, t|\mathbf{r}_0, t_0)\} \\ &= I_{nm}G_{nm}(\mathbf{r}_1, t_1|\mathbf{r}_0, t_0) - I_{nm}G_{nm}(\mathbf{r}_0, -t_0|\mathbf{r}_1, -t_1). \end{aligned} \quad (5.3.22)$$

To rewrite the first term in equation (5.3.22), we will use *Gauss' Theorem*, stating:

$$\int_V \nabla \cdot \mathbf{F}(\mathbf{r}, t) d^3r = \oint_S \mathbf{F}(\mathbf{r}, t) \cdot d\mathbf{A} \quad (5.3.23)$$

$\mathbf{F}(\mathbf{r}, t)$ is some vector field, and the integration runs over some volume V enclosed by a boundary surface S . $\oint_S d\mathbf{A}$ is thus a surface integral over the boundary surface S .

We may construct a vector field, from two scalar fields $U(\mathbf{r}, t)$ and $V(\mathbf{r}, t)$ as

$$U(\mathbf{r}, t)\nabla V(\mathbf{r}, t) - V(\mathbf{r}, t)\nabla U(\mathbf{r}, t). \quad (5.3.24)$$

Using the following identity[§]:

$$\nabla \cdot (U(\mathbf{r}, t) \nabla V(\mathbf{r}, t)) = \nabla U(\mathbf{r}, t) \cdot \nabla V(\mathbf{r}, t) + U(\mathbf{r}, t) \nabla^2 V(\mathbf{r}, t), \quad (5.3.25)$$

Gauss' theorem (5.3.23) may be written in the appropriate form:

$$\int [U(\mathbf{r}, t) \nabla^2 V(\mathbf{r}, t) - V(\mathbf{r}, t) \nabla^2 U(\mathbf{r}, t)] d^3 r = \oint [U(\mathbf{r}, t) \nabla V(\mathbf{r}, t) - V(\mathbf{r}, t) \nabla U(\mathbf{r}, t)] \cdot d\mathbf{A} \quad (5.3.26)$$

Using this result, we may rewrite the first term of equation (5.3.22), and find:

$$\begin{aligned} & \iint_{-\infty}^{t_1^+} d^3 r dt - c^2 \{ G_{nm}^+(\mathbf{r}, t | \mathbf{r}_0, t_0) \nabla^2 G_{nm}^+(\mathbf{r}, -t | \mathbf{r}_1, -t_1) - G_{nm}^+(\mathbf{r}, -t | \mathbf{r}_1, -t_1) \nabla^2 G_{nm}^+(\mathbf{r}, t | \mathbf{r}_0, t_0) \} \\ &= - \int_{-\infty}^{t_1^+} dt \oint c^2 \{ G_{nm}^+(\mathbf{r}, t | \mathbf{r}_0, t_0) \nabla G_{nm}^+(\mathbf{r}, -t | \mathbf{r}_1, -t_1) - G_{nm}^+(\mathbf{r}, -t | \mathbf{r}_1, -t_1) \nabla G_{nm}^+(\mathbf{r}, t | \mathbf{r}_0, t_0) \} \cdot d\mathbf{A} \end{aligned} \quad (5.3.27)$$

Letting $\bar{G}^+(\mathbf{r}, t | \mathbf{r}', t')$ vanish at the boundary, we find the integration to vanish, and we may conclude from equation (5.3.22):

$$\bar{G}^+(\mathbf{r}_1, t_1 | \mathbf{r}_0, t_0) = \bar{G}^+(\mathbf{r}_0, -t_0 | \mathbf{r}_1, -t_1) \quad (5.3.28)$$

From this result we may derive the *reciprocal* equation for the Green's function. The general equation for the Green's function is:

$$\left(2i\omega_L \frac{\partial}{\partial t_0} - c^2 \nabla_0^2 - \omega_L^2 \right) \bar{G}^+(\mathbf{r}_0, t_0 | \mathbf{r}, t) = \bar{I} \delta(\mathbf{r} - \mathbf{r}_0) \delta(t - t_0) \quad (5.3.29)$$

Making the transformations $t_0 \rightarrow -t_0$ and $t \rightarrow -t$ and using the result stated in equation (5.3.28), we finally arrive at the *reciprocal equation* for the Green's function.

$$\left(-2i\omega_L \frac{\partial}{\partial t_0} - c^2 \nabla_0^2 - \omega_L^2 \right) \bar{G}^+(\mathbf{r}, t | \mathbf{r}_0, t_0) = \bar{I} \delta(\mathbf{r} - \mathbf{r}_0) \delta(t - t_0) \quad (5.3.30)$$

We are now ready to derive the general solution for our equations of motion for the system (5.3.6).

[§]Derivation of this has been put in Appendix C.8

5.3.2 The general solution of the Diffusion Equation

Let us recall the diffusion equation we are working with (5.3.6).

$$\left(2i\omega_L \frac{\partial}{\partial t'} - c^2 \nabla'^2 - \omega_L^2\right) \psi^+(\mathbf{r}', t') = \rho^+(\mathbf{r}', t) \quad (5.3.31)$$

(Primed and unprimed variables are chosen by hindsight) The second equation we need, in order to derive a solution, is the reciprocal equation (5.3.30)

$$\left(-2i\omega_L \frac{\partial}{\partial t'} - c^2 \nabla'^2 - \omega_L^2\right) \bar{G}^+(\mathbf{r}, t|\mathbf{r}', t') = \bar{I}\delta(\mathbf{r} - \mathbf{r}')\delta(t - t') \quad (5.3.32)$$

Next we make the inner product[¶] of equation (5.3.31) with $\bar{G}^+(\mathbf{r}, t|\mathbf{r}', t')$ from the left, and inner product of equation (5.3.32) with $\psi^+(\mathbf{r}', t')$ from the right. We further more integrate over the volume bounded by the boundary sphere, and also over time $t \in]t_0, t^+[$, where we understand $t^+ = t + \epsilon$. To make the derivation explicit we will consider the equation component wise. This will also clarify what we mean by the products from left and right. Doing this we arrive at the following two equations:

$$\begin{aligned} \iint_{t_0}^{t^+} d^3 r' dt' G_{rs}^+(\mathbf{r}, t|\mathbf{r}', t') \rho_s^+(\mathbf{r}', t) \\ = \iint_{t_0}^{t^+} d^3 r' dt' G_{rs}^+(\mathbf{r}, t|\mathbf{r}', t') \left(2i\omega_L \frac{\partial}{\partial t'} - c^2 \nabla'^2 - \omega_L^2\right) \psi_s^+(\mathbf{r}', t') \end{aligned} \quad (5.3.33a)$$

$$\psi_r^+(\mathbf{r}, t) = \iint_{t_0}^{t^+} d^3 r' dt' \psi_s^+(\mathbf{r}', t') \left(2i\omega_L \frac{\partial}{\partial t'} - c^2 \nabla'^2 - \omega_L^2\right) G_{rs}^+(\mathbf{r}, t|\mathbf{r}', t') \quad (5.3.33b)$$

We then subtract the two equations (5.3.33a) and (5.3.33b), and yield:

$$\begin{aligned} \psi_r^+(\mathbf{r}, t) - \iint_{t_0}^{t^+} d^3 r' dt' G_{rs}(\mathbf{r}, t|\mathbf{r}', t') \rho_s^+(\mathbf{r}', t) = \\ \iint_{t_0}^{t^+} d^3 r' dt' \left\{ -2i\omega_L \left(\psi_s^+(\mathbf{r}', t') \frac{\partial}{\partial t'} G_{rs}^+(\mathbf{r}, t|\mathbf{r}', t') + G_{rs}^+(\mathbf{r}, t|\mathbf{r}', t') \frac{\partial}{\partial t'} \psi_s^+(\mathbf{r}', t')\right) \right. \end{aligned} \quad (5.3.34a)$$

$$\left. - c^2 \left(\psi_s^+(\mathbf{r}', t') \nabla'^2 G_{rs}^+(\mathbf{r}, t|\mathbf{r}', t') - G_{rs}^+(\mathbf{r}, t|\mathbf{r}', t') \nabla'^2 \psi_s^+(\mathbf{r}', t')\right) \right. \quad (5.3.34b)$$

$$\left. - \omega_L \left(\psi_s^+(\mathbf{r}', t') G_{rs}^+(\mathbf{r}, t|\mathbf{r}', t') - G_{rs}^+(\mathbf{r}, t|\mathbf{r}', t') \psi_s^+(\mathbf{r}', t')\right) \right\} \quad (5.3.34c)$$

[¶]Here we mean the usual dot product $\mathbf{r} \cdot \mathbf{k} = r_n k_n$, using Einstein notation

The term (5.3.34c) vanish trivially. Considering the term (5.3.34a) we write this as:

$$\begin{aligned} \iint_{t_0}^{t^+} d^3 r' dt' - 2i\omega_L \frac{\partial}{\partial t'} (G_{rs}^+(\mathbf{r}, t | \mathbf{r}', t') \psi_s^+(\mathbf{r}', t')) \\ = - \int d^3 r' 2i\omega_L [G_{rs}^+(\mathbf{r}, t | \mathbf{r}', t') \psi_s^+(\mathbf{r}', t')]_{t_0}^{t^+} \end{aligned}$$

From equation (5.3.17) we see that the upper limit vanish, since $t < t^+$. In this respect we will write the term (5.3.34a) on vector form, as the following.

$$2i\omega_L \int d^3 r' \bar{G}^+(\mathbf{r}, t | \mathbf{r}', t_0) \cdot \psi^+(\mathbf{r}', t_0) \quad (5.3.35)$$

We now only need to consider the term (5.3.34b). We will consider this term on vector form, as it will clarify the next steps. In order to manipulate this term we need a variation of Gauss' theorem concerning vectors.

$$\begin{aligned} \int [\mathbf{E} \cdot \nabla^2 \mathbf{F} - \mathbf{F} \cdot \nabla^2 \mathbf{E}] dV \\ = \oint \left\{ [\mathbf{E} \nabla \cdot \mathbf{F} - \mathbf{F} \nabla \cdot \mathbf{E}] \cdot \mathbf{n} - [\mathbf{E} \cdot (\mathbf{n} \times \nabla \times \mathbf{F}) + \nabla \times \mathbf{E} \cdot (\mathbf{n} \times \mathbf{F})] \right\} dA, \end{aligned} \quad (5.3.36)$$

where we have used $d\mathbf{A} = \mathbf{n}dA$. To shorten the notation we suppressed space and time dependence. The derivation of the equation has been put in Appendix C.8.1.

We want to bring the term (5.3.34b) to the form of a surface integral. To do that we use the explicit expression for the Green's function, derived in equation (5.3.16). Inserting these into equation (5.3.34b) leaves us with:

$$-c^2 \iint_{t_0}^{t^+} d^3 r' dt' \sum_n \frac{1}{\omega_n^2} (\psi^+(\mathbf{r}', t') \cdot \nabla'^2 \mathbf{F}_n(\mathbf{r}', t') - \nabla'^2 \psi^+(\mathbf{r}', t') \cdot \mathbf{F}_n(\mathbf{r}', t')) \mathbf{F}_n(\mathbf{r}, t) \quad (5.3.37)$$

The rewriting is not as trivial as one might think, some tensor considerations are needed, see Appendix C.7. Using *Gauss' theorem* equation (5.3.36), equation (5.3.37) can be written as the

following surface integral.

$$\begin{aligned}
& -c^2 \sum_n \int_{t_0}^{t^+} dt' \oint dA \frac{1}{\omega_n^2} \left\{ \left[\boldsymbol{\psi}^+(\mathbf{r}', t') \boldsymbol{\nabla}' \cdot \mathbf{F}_n(\mathbf{r}', t') - \mathbf{F}_n(\mathbf{r}', t') \boldsymbol{\nabla}' \cdot \boldsymbol{\psi}^+(\mathbf{r}', t') \right] \cdot \mathbf{n} \right. \\
& \left. - \left[\boldsymbol{\psi}^+(\mathbf{r}', t') \cdot (\mathbf{n} \times \boldsymbol{\nabla}' \times \mathbf{F}_n(\mathbf{r}', t')) + \boldsymbol{\nabla}' \times \boldsymbol{\psi}^+(\mathbf{r}', t') \cdot (\mathbf{n} \times \mathbf{F}_n(\mathbf{r}', t')) \right] \right\} \mathbf{F}_n(\mathbf{r}, t) \\
& = -c^2 \sum_n \int_{t_0}^{t^+} dt' \oint dA \frac{1}{\omega_n^2} \left\{ (\boldsymbol{\psi}^+(\mathbf{r}', t') \cdot \mathbf{n})(\boldsymbol{\nabla}' \cdot \mathbf{F}_n(\mathbf{r}', t')) - (\boldsymbol{\nabla}' \cdot \boldsymbol{\psi}^+(\mathbf{r}', t'))(\mathbf{n} \cdot \mathbf{F}_n(\mathbf{r}', t')) \right. \\
& \left. - \left[\boldsymbol{\psi}^+(\mathbf{r}', t') \cdot (\mathbf{n} \times \boldsymbol{\nabla}' \times \mathbf{F}_n(\mathbf{r}', t')) + \boldsymbol{\nabla}' \times \boldsymbol{\psi}^+(\mathbf{r}', t') \cdot (\mathbf{n} \times \mathbf{F}_n(\mathbf{r}', t')) \right] \right\} \mathbf{F}_n(\mathbf{r}, t) \quad (5.3.38)
\end{aligned}$$

Again using the expression for the Green's function, (5.3.16), we write the term (5.3.38) to the following:

$$\begin{aligned}
& -c^2 \int_{t_0}^{t^+} dt' \oint dA \left\{ (\boldsymbol{\psi}^+(\mathbf{r}', t') \cdot \mathbf{n})(\boldsymbol{\nabla}' \cdot \bar{\bar{G}}^+(\mathbf{r}, t|\mathbf{r}', t')) - (\boldsymbol{\nabla}' \cdot \boldsymbol{\psi}^+(\mathbf{r}', t'))(\mathbf{n} \cdot \bar{\bar{G}}^+(\mathbf{r}, t|\mathbf{r}', t')) \right. \\
& \left. - \left[\boldsymbol{\psi}^+(\mathbf{r}', t') \cdot (\mathbf{n} \times \boldsymbol{\nabla}' \times \bar{\bar{G}}^+(\mathbf{r}, t|\mathbf{r}', t')) + \boldsymbol{\nabla}' \times \boldsymbol{\psi}^+(\mathbf{r}', t') \cdot (\mathbf{n} \times \bar{\bar{G}}^+(\mathbf{r}, t|\mathbf{r}', t')) \right] \right\} \quad (5.3.39)
\end{aligned}$$

Using the result of equation (5.3.39) and equation (5.3.35) we may rewrite equation (5.3.34) to the following:

$$\begin{aligned}
\boldsymbol{\psi}^+(\mathbf{r}, t) & = 2i\omega_L \int d^3r' \bar{\bar{G}}^+(\mathbf{r}, t|\mathbf{r}', t_0) \cdot \boldsymbol{\psi}^+(\mathbf{r}', t_0) + \iint_{t_0}^t d^3r' dt' \bar{\bar{G}}^+(\mathbf{r}, t|\mathbf{r}', t') \cdot \boldsymbol{\rho}^+(\mathbf{r}', t) \\
& - c^2 \int_{t_0}^{t^+} dt' \oint dA \left\{ (\boldsymbol{\psi}^+(\mathbf{r}', t') \cdot \mathbf{n})(\boldsymbol{\nabla}' \cdot \bar{\bar{G}}^+(\mathbf{r}, t|\mathbf{r}', t')) - (\boldsymbol{\nabla}' \cdot \boldsymbol{\psi}^+(\mathbf{r}', t'))(\mathbf{n} \cdot \bar{\bar{G}}^+(\mathbf{r}, t|\mathbf{r}', t')) \right. \\
& \left. - \left[\boldsymbol{\psi}^+(\mathbf{r}', t') \cdot (\mathbf{n} \times \boldsymbol{\nabla}' \times \bar{\bar{G}}^+(\mathbf{r}, t|\mathbf{r}', t')) + \boldsymbol{\nabla}' \times \boldsymbol{\psi}^+(\mathbf{r}', t') \cdot (\mathbf{n} \times \bar{\bar{G}}^+(\mathbf{r}, t|\mathbf{r}', t')) \right] \right\} \quad (5.3.40)
\end{aligned}$$

The first term in equation (5.3.40) represent the freely evolving solution for a given set of initial conditions. The second term represent the effect of some perturbation of the evolution of the system due to some source $\boldsymbol{\rho}^+(\mathbf{r}, t)$. The last terms represent the effect of boundary conditions. In the case where the boundary conditions are given such that the fields as well as Green's functions vanish at the boundary, we could put the boundary at infinity, the last four terms in equation (5.3.40) will vanish and we are left with:

$$\psi^+(\mathbf{r}, t) = 2i\omega_L \int d^3 r' \bar{G}^+(\mathbf{r}, t|\mathbf{r}', t_0) \cdot \psi^+(\mathbf{r}', t_0) + \iint_{t_0}^t d^3 r' dt' \bar{G}^+(\mathbf{r}, t|\mathbf{r}', t') \cdot \rho^+(\mathbf{r}', t) \quad (5.3.41)$$

Returning to the negatively oscillating part of the field, we find the relevant Green's function to be defined by:

$$\left(-2i\omega_L \frac{\partial}{\partial t} - \omega_L^2 - c^2 \nabla^2\right) \bar{G}^-(\mathbf{r}, t|\mathbf{r}', t') = \bar{I} \delta(\mathbf{r} - \mathbf{r}') \delta(t - t') \quad (5.3.42)$$

However taking complex conjugation on equation (5.3.8) leaves us with

$$\left(-2i\omega_L \frac{\partial}{\partial t} - \omega_L^2 - c^2 \nabla^2\right) (\bar{G}^+(\mathbf{r}, t|\mathbf{r}', t'))^* = \bar{I} \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'), \quad (5.3.43)$$

thus we may choose $\bar{G}^-(\mathbf{r}, t|\mathbf{r}', t')$ such that:

$$\bar{G}^-(\mathbf{r}, t|\mathbf{r}', t') = (\bar{G}^+(\mathbf{r}, t|\mathbf{r}', t'))^*. \quad (5.3.44)$$

The solution to equation (5.3.6) regarding the negative frequency component thus reads:

$$\psi^-(\mathbf{r}, t) = -2i\omega_L \int d^3 r' \bar{G}^-(\mathbf{r}, t|\mathbf{r}', t_0) \cdot \psi^-(\mathbf{r}', t_0) + \iint_{t_0}^t d^3 r' dt' \bar{G}^-(\mathbf{r}, t|\mathbf{r}', t') \cdot \rho^-(\mathbf{r}', t) \quad (5.3.45)$$

On this form, the solution seem enticingly simple, one might even have guessed it. In the preceding sections we showed how to treat initial and boundary conditions for the particular three dimensional diffusion equation. Though we neglected the boundary terms, there was no way we could have guessed this general solution. Consider the solution for the slightly different three dimensional wave equation.^{||} Again we neglect boundary terms.

$$\begin{aligned} \psi^-(\mathbf{r}, t) = & - \int d^3 r' \left[\left(\frac{d}{dt'} \bar{G}^-(\mathbf{r}, t|\mathbf{r}', t') \right) \cdot \psi^-(\mathbf{r}', t') - \bar{G}^-(\mathbf{r}, t|\mathbf{r}', t') \cdot \left(\frac{d}{dt'} \psi^-(\mathbf{r}', t') \right) \right]_{t'=0} \\ & + \iint_{t_0}^t d^3 r' dt' \bar{G}^-(\mathbf{r}, t|\mathbf{r}', t') \cdot \rho^-(\mathbf{r}', t) \end{aligned} \quad (5.3.46)$$

The initial conditions are treated quite different in this solution, and we would have to know time derivative of both the Green's function and the field at initial time. We thus conclude this

^{||}This is found in (47) pg 1790

discussion by noting the importance of treating differential equations with care. *Watching an accelerometer in a car does not tell us where the car is, or where it is headed.*

We are now ready to return to the quantum mechanical equations of motion for field and spin, (5.1.3) and (5.2.1). Using Green's functions we will derive solutions to these equations. First we bring the equations to the slowly oscillating form.

5.4 Slowly varying operator equations

Using the expansion (4.2.40) we find the set of differential equations concerning slowly oscillating fields, derived from equation (5.1.11). The calculations are similar to equation (5.3.3). Using this we arrive at the following equation concerning the positively oscillating field component. We will not write the negatively oscillating field component since this can be found by hermitian conjugating the positively oscillating component.

$$\begin{aligned} & \left(\left(\frac{\partial}{\partial t} + i\omega_L \right)^2 - c^2 \nabla^2 \epsilon^{-1}(\mathbf{r}) \right) \tilde{\mathbf{D}}^+(\mathbf{r}, t) \\ &= \int d^3 r' c^2 \beta \nabla \times \nabla \times \bar{\delta}^T(\mathbf{r}, \mathbf{r}') \cdot \left\{ \frac{1}{2} \lambda(\mathbf{r}') \tilde{\mathbf{D}}^+(\mathbf{r}', t) + i \hat{\mathbf{J}}(\mathbf{r}') \times \tilde{\mathbf{D}}^+(\mathbf{r}', t) \right\} \end{aligned} \quad (5.4.1)$$

Making the slowly varying approximation done in equation (5.3.5) we finally arrive at the following wave equation:

$$\begin{aligned} & \left(2i\omega_L \frac{\partial}{\partial t} - \omega_L^2 - c^2 \nabla^2 \epsilon^{-1}(\mathbf{r}) \right) \tilde{\mathbf{D}}^+(\mathbf{r}, t) \\ &= \int d^3 r' c^2 \beta \nabla \times \nabla \times \bar{\delta}^T(\mathbf{r}, \mathbf{r}') \cdot \left\{ \frac{1}{2} \lambda(\mathbf{r}') \tilde{\mathbf{D}}^+(\mathbf{r}', t) + i \hat{\mathbf{J}}(\mathbf{r}') \times \tilde{\mathbf{D}}^+(\mathbf{r}', t) \right\} \end{aligned} \quad (5.4.2)$$

These equations differ from the ones in equation (5.3.6) by the factor of $\epsilon^{-1}(\mathbf{r})$, if however we assume that $\epsilon^{-1}(\mathbf{r})$ changes slowly compared to $\tilde{\mathbf{D}}^\pm(\mathbf{r}, t)$ we may use the solutions found from the equations (5.3.6). This is said knowing that the type of physical propagations we will consider in the gas, are happening on a small scale, far from the edges of the gas. In the following final part of this chapter we find the general solutions to equations (5.4.2).

5.4.1 General solution using Green's function

Using equation (5.3.41) we find the positively oscillating components of the quantized displaced electric field to be given by:

$$\begin{aligned} \tilde{\mathbf{D}}^+(\mathbf{r}, t) = & 2i\omega_L \int d^3 r' \bar{G}^+(\mathbf{r}, t|\mathbf{r}', t_0) \cdot \tilde{\mathbf{D}}^+(\mathbf{r}', t_0) + \iint_{t_0}^t d^3 r' dt' \bar{G}^+(\mathbf{r}, t|\mathbf{r}', t') \\ & \cdot \int d^3 r'' c^2 \beta \nabla' \times \nabla' \times \bar{\delta}^T(\mathbf{r}', \mathbf{r}'') \cdot \left\{ \frac{1}{2} \lambda(\mathbf{r}'') \tilde{\mathbf{D}}^+(\mathbf{r}'', t') + i \hat{\mathbf{J}}(\mathbf{r}'', t') \times \tilde{\mathbf{D}}^+(\mathbf{r}'', t') \right\} \end{aligned} \quad (5.4.3)$$

Let us consider the last integral of equation (5.4.3). Using partial integration twice in the variable \mathbf{r}' we find:

$$\begin{aligned} & \iint_{t_0}^t d^3 r' dt' \nabla' \times \nabla' \times \bar{G}^+(\mathbf{r}, t|\mathbf{r}', t') \\ & \cdot \int d^3 r'' c^2 \beta \bar{\delta}^T(\mathbf{r}', \mathbf{r}'') \cdot \left\{ \frac{1}{2} \lambda(\mathbf{r}'') \tilde{\mathbf{D}}^+(\mathbf{r}'', t') + i \hat{\mathbf{J}}(\mathbf{r}'', t') \times \tilde{\mathbf{D}}^+(\mathbf{r}'', t') \right\} \end{aligned}$$

Since the tensor $\nabla' \times \nabla' \times \bar{G}^+(\mathbf{r}, t|\mathbf{r}', t')$ is transverse, we may do the integration over \mathbf{r}' . We then find:

$$\iint_{t_0}^t d^3 r' dt' \nabla' \times \nabla' \times \bar{G}^+(\mathbf{r}, t|\mathbf{r}', t') \cdot c^2 \beta \left\{ \frac{1}{2} \lambda(\mathbf{r}') \tilde{\mathbf{D}}^+(\mathbf{r}', t') + i \hat{\mathbf{J}}(\mathbf{r}', t') \times \tilde{\mathbf{D}}^+(\mathbf{r}', t') \right\} \quad (5.4.4)$$

Let us also define the freely propagating field only subject to initial conditions:

$$\tilde{\mathbf{D}}_0^+(\mathbf{r}, t) = 2i\omega_L \int d^3 r' \bar{G}^+(\mathbf{r}, t|\mathbf{r}', t_0) \cdot \tilde{\mathbf{D}}^+(\mathbf{r}', t_0) \quad (5.4.5)$$

Similar calculations apply to the negatively oscillating part of the quantized displaced electric field. Using definitions (5.4.5) the equation (5.4.3), may appear in a shorter form. We have written the negatively oscillating part of the displaced electric field as well.

$$\begin{aligned} \tilde{\mathbf{D}}^+(\mathbf{r}, t) = & \tilde{\mathbf{D}}_0^+(\mathbf{r}, t) \\ & + \iint_{t_0}^t d^3 r' dt' \nabla' \times \nabla' \times \bar{G}^+(\mathbf{r}, t|\mathbf{r}', t') \cdot c^2 \beta \left\{ \frac{1}{2} \lambda(\mathbf{r}') \tilde{\mathbf{D}}^+(\mathbf{r}', t') + i \hat{\mathbf{J}}(\mathbf{r}', t') \times \tilde{\mathbf{D}}^+(\mathbf{r}', t') \right\} \end{aligned} \quad (5.4.6)$$

$$\begin{aligned} \tilde{\mathbf{D}}^-(\mathbf{r}, t) = & \tilde{\mathbf{D}}_0^-(\mathbf{r}, t) \\ & + \iint_{t_0}^t d^3r' dt' \nabla' \times \nabla' \times \bar{\bar{G}}^-(\mathbf{r}, t | \mathbf{r}', t') \cdot c^2 \beta \left\{ \frac{1}{2} \lambda(\mathbf{r}') \tilde{\mathbf{D}}^-(\mathbf{r}', t') - i \hat{\mathbf{J}}(\mathbf{r}', t') \times \tilde{\mathbf{D}}^-(\mathbf{r}', t') \right\}. \end{aligned} \quad (5.4.7)$$

These equations are by all means ideally suited perturbative treatment. We are dealing with small and slowly varying operators in an integral. Along with the equation regarding spin,

$$\frac{d}{dt} \hat{\mathbf{J}}(\mathbf{r}, t) = \frac{2i\beta}{\hbar\epsilon_0} \hat{\mathbf{J}}(\mathbf{r}, t) \times \tilde{\mathbf{D}}^+(\mathbf{r}, t) \times \tilde{\mathbf{D}}^-(\mathbf{r}, t), \quad (5.4.8)$$

these equations constitutes the coupled set of differential and integral equations, describing our system. In the spin equation we used the field representation of the spin, defined in equation (5.1.5). The spin equation has an integral representation, which follows from fundamental integral theory.

$$\hat{\mathbf{J}}(\mathbf{r}, t) = \hat{\mathbf{J}}_0(\mathbf{r}) + \int_{t_0}^t dt' \frac{2i\beta}{\hbar\epsilon_0} \hat{\mathbf{J}}(\mathbf{r}, t') \times \tilde{\mathbf{D}}^+(\mathbf{r}, t') \times \tilde{\mathbf{D}}^-(\mathbf{r}, t'), \quad (5.4.9)$$

where we used $\hat{\mathbf{J}}(\mathbf{r}, t_0) = \hat{\mathbf{J}}_0(\mathbf{r})$. In principle we know all evolution of our system, provided we know the Green's function. What complicates the problem considerably is the fact that the source driving the system is depending in a nontrivial way on the fields we are trying to find. This leaves us in a position where we must approximate the solution, in order to find anything analytical. In the following part of this thesis we will do the approximations allowing us to solve the integralequations (5.4.6), (5.4.7) and (5.4.9).

Part III

Solving the Equations

Spatial Mean Theory

In this chapter we formally solve the integral equations (5.4.6), (5.4.7) and (5.4.9). Our line of approach will firstly be to make the *Born Approximation* on the equations, and consider the two lowest order terms. Secondly we will, since we in this approximation know nothing about the location of the atoms, take the spatial mean over positions of atoms. This introduces some correlations, when considering second order terms. Thirdly we will calculate Green's functions, both in the case of physical light propagation, and also for terms introduced by density and spin correlation functions. Finally we write out the general evolution of the spin of the atoms. Our main concern in this thesis is to establish a consistent three dimensional theory describing light matter interaction, to test its utility we consider the atoms in particular. Future work will deal with the displaced electric field as well.

6.1 Born Approximation

Let us consider a general integral equation describing weak scattering, the variable λ is considered small.

$$\begin{aligned} \psi(\mathbf{r}, t) = & \int d^3 r' \bar{G}(\mathbf{r}, t | \mathbf{r}', t') \psi(\mathbf{r}', t_0) \\ & + \iint d^3 r' dt' \bar{G}(\mathbf{r}, t | \mathbf{r}', t') \cdot \iint d^3 r'' dt'' \lambda \bar{V}(\mathbf{r}', t' | \mathbf{r}'', t'') \cdot \psi(\mathbf{r}'', t''), \end{aligned} \quad (6.1.1)$$

where $\bar{G}(\mathbf{r}, t|\mathbf{r}', t')$ is the free propagator between scattering events. $\bar{V}(\mathbf{r}, t|\mathbf{r}', t')$ represents scattering. Using a short operator notation this may be written as:

$$\psi = G\psi_0 + \lambda GV\psi \quad (6.1.2)$$

We may write this equation as an infinite sum, in the free solution ψ_0 , by iteratively inserting ψ . Provided λ small this sum converges.

$$\psi = G\psi_0 + \lambda GV\psi_0 + \lambda^2 GVG\psi_0 + \lambda^3 GVGVG\psi_0 + \lambda^4 GVGVGVG\psi_0 + \dots \quad (6.1.3)$$

We only wanted to consider the solution to second order in the small expansion parameter λ , and hence,

$$\psi = G\psi_0 + \lambda GV\psi_0 + \lambda^2 GVG\psi_0. \quad (6.1.4)$$

We will use this expansion on our integral equations (5.4.6), (5.4.7) and (5.4.9). In our case we note that we are dealing with two type of scattering processes, one type where light merely bounces off of the atoms, and one where the light interacts with the atoms in the way $\hat{\mathbf{J}}(\mathbf{r}, t) \times$. The second order solution to the field $\tilde{\mathbf{D}}^+(\mathbf{r}, t)$ may then be written as:

$$\begin{aligned} \tilde{\mathbf{D}}^+(\mathbf{r}, t) = & \tilde{\mathbf{D}}_0^+(\mathbf{r}, t) \\ & + \iint_{t_0}^t d^3 r' dt' \nabla' \times \nabla' \times \bar{G}^+(\mathbf{r}, t|\mathbf{r}', t') \cdot c^2 \beta \left\{ \frac{1}{2} \lambda(\mathbf{r}') \tilde{\mathbf{D}}_0^+(\mathbf{r}', t') + i \hat{\mathbf{J}}_0(\mathbf{r}') \times \tilde{\mathbf{D}}_0^+(\mathbf{r}', t') \right\} \\ & + \iint_{t_0}^t d^3 r' dt' \nabla' \times \nabla' \times \bar{G}^+(\mathbf{r}, t|\mathbf{r}', t') \cdot c^2 \beta \left\{ \frac{1}{2} \lambda(\mathbf{r}') \left[\iint_{t_0}^{t'} d^3 r'' dt'' \nabla'' \times \nabla'' \times \bar{G}^+(\mathbf{r}', t'|\mathbf{r}'', t'') \cdot \right. \right. \\ & \left. \left. c^2 \beta \left\{ \frac{1}{2} \lambda(\mathbf{r}'') \tilde{\mathbf{D}}_0^+(\mathbf{r}'', t'') + i \hat{\mathbf{J}}_0(\mathbf{r}'') \times \tilde{\mathbf{D}}_0^+(\mathbf{r}'', t'') \right\} \right] + i \hat{\mathbf{J}}_0(\mathbf{r}') \times \tilde{\mathbf{D}}_0^+(\mathbf{r}', t') \right\} \\ & + \iint_{t_0}^t d^3 r' dt' \nabla' \times \nabla' \times \bar{G}^+(\mathbf{r}, t|\mathbf{r}', t') \cdot c^2 \beta \left\{ \frac{1}{2} \lambda(\mathbf{r}') \tilde{\mathbf{D}}_0^+(\mathbf{r}', t') + i \hat{\mathbf{J}}_0(\mathbf{r}') \times \right. \\ & \left. \left[\iint_{t_0}^{t'} d^3 r'' dt'' \nabla'' \times \nabla'' \times \bar{G}^+(\mathbf{r}', t'|\mathbf{r}'', t'') \cdot c^2 \beta \left\{ \frac{1}{2} \lambda(\mathbf{r}'') \tilde{\mathbf{D}}_0^+(\mathbf{r}'', t'') + i \hat{\mathbf{J}}_0(\mathbf{r}'') \times \tilde{\mathbf{D}}_0^+(\mathbf{r}'', t'') \right\} \right] \right\} \\ & + \iint_{t_0}^t d^3 r' dt' \nabla' \times \nabla' \times \bar{G}^+(\mathbf{r}, t|\mathbf{r}', t') \cdot c^2 \beta \left\{ \frac{1}{2} \lambda(\mathbf{r}') \tilde{\mathbf{D}}_0^+(\mathbf{r}', t') \right. \\ & \left. + i \left[\int_{t_0}^{t'} dt'' \frac{2i\beta}{\hbar \epsilon_0} \hat{\mathbf{J}}_0(\mathbf{r}') \times \tilde{\mathbf{D}}_0^+(\mathbf{r}', t'') \times \tilde{\mathbf{D}}_0^-(\mathbf{r}', t'') \right] \times \tilde{\mathbf{D}}_0^+(\mathbf{r}', t') \right\} \quad (6.1.5) \end{aligned}$$

This equation can be represented in a Feynman-like diagram. The equation as it is written above can be slightly difficult to read and interpret. In the diagram Figure 6.1 we mark an interaction corresponding to light bouncing of the atom, with a black circle. A vertical bar thus represents the nontrivial scattering process. The zeroth order term is the case where light passes through the sample without scattering. First order terms describe single scatter effects, that light exiting the sample only interacted once with the atoms. Second order terms cover the case where an exiting photon interacted with two atoms, but also the case where an exiting photon interacted with an atom, previously involved in a scattering event. A similar expansion apply for the negative

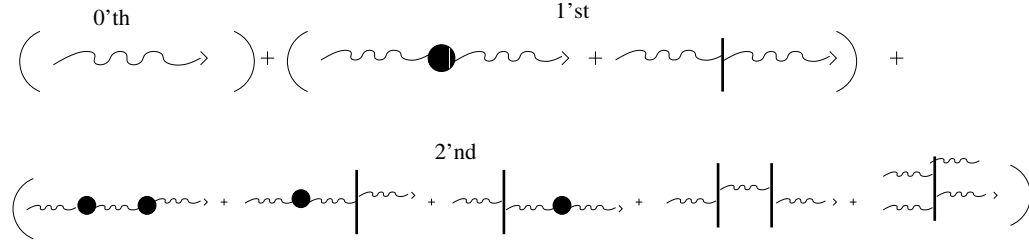


Figure 6.1: Diagrammatic representation of field expansion. Bars represent light- atomic spin interaction, and spheres represents light bouncing of atoms.

frequency component, our focus being the spin equations, we will not write it out. Turning to the spin equation (5.4.8), and expanding to second order in β gives the following:

$$\begin{aligned}
 \frac{d}{dt} \hat{\mathbf{J}}(\mathbf{r}, t) = & \frac{2i\beta}{\hbar\epsilon_0} \hat{\mathbf{J}}_0(\mathbf{r}) \times \tilde{\mathbf{D}}_0^+(\mathbf{r}, t) \times \tilde{\mathbf{D}}_0^-(\mathbf{r}, t) + \frac{2i\beta}{\hbar\epsilon_0} \hat{\mathbf{J}}_0(\mathbf{r}) \times \left[\iint_{t_0}^t d^3 r' dt' \nabla' \times \nabla' \times \bar{\bar{G}}^+(\mathbf{r}, t | \mathbf{r}', t') \cdot \right. \\
 & \left. c^2 \beta \left\{ \frac{1}{2} \lambda(\mathbf{r}') \tilde{\mathbf{D}}_0^+(\mathbf{r}', t') + i \hat{\mathbf{J}}_0(\mathbf{r}') \times \tilde{\mathbf{D}}_0^+(\mathbf{r}', t') \right\} \right] \times \tilde{\mathbf{D}}_0^-(\mathbf{r}, t) \\
 & + \frac{2i\beta}{\hbar\epsilon_0} \hat{\mathbf{J}}_0(\mathbf{r}) \times \tilde{\mathbf{D}}_0^+(\mathbf{r}, t) \times \left[\iint_{t_0}^t d^3 r' dt' \nabla' \times \nabla' \times \bar{\bar{G}}^-(\mathbf{r}, t | \mathbf{r}', t') \cdot \right. \\
 & \left. c^2 \beta \left\{ \frac{1}{2} \lambda(\mathbf{r}') \tilde{\mathbf{D}}_0^-(\mathbf{r}', t') - i \hat{\mathbf{J}}_0(\mathbf{r}') \times \tilde{\mathbf{D}}_0^-(\mathbf{r}', t') \right\} \right] \\
 & + \frac{2i\beta}{\hbar\epsilon_0} \left[\int_{t_0}^t dt' \frac{2i\beta}{\hbar\epsilon_0} \hat{\mathbf{J}}_0(\mathbf{r}) \times \tilde{\mathbf{D}}_0^+(\mathbf{r}, t') \times \tilde{\mathbf{D}}_0^-(\mathbf{r}, t') \right] \times \tilde{\mathbf{D}}_0^+(\mathbf{r}, t) \times \tilde{\mathbf{D}}_0^-(\mathbf{r}, t) \quad (6.1.6)
 \end{aligned}$$

As we did it with equation (6.1.5), we will also represent equation (6.1.6) in a Feynman like diagram, Figure 6.2. Since we are considering some atom, or some point in space where there might be an atom, the diagrams can be read such that the longest vertical bar represents the atom in consideration, on the left side is depicted scattering events for the incoming photon, and on the right side, scattering processes concerning the outgoing photon. As we will calculate, some of these diagrams cancels, and only physically meaningful terms remain. Having put our equation to the form (6.1.6) brings us very close to a solution. However this step forward unfortunately

Figure 6.2: Equation of motion for the spin of the atoms represented diagrammatically.

brought us one step backwards as well. The spin operator $\hat{\mathbf{J}}(\mathbf{r}, t)$ being a physical observable, has to be hermitian. That means that the equation (6.1.6) has to be hermitian. However the complexity of the equation leaves us with little chance of checking this property. What makes the question even more complex is the introduction of commutation relations, the individual parts of each term do not commute in all of time and space. What one will find if naively considering each term alone, is that they are not independently hermitian, thus the rate of success depends crucially on our ability to find matching terms. The problem arise in the term describing a photon interacting with spin of one atom located at position \mathbf{r}' , and then interacting with the spin of a second atom located at position \mathbf{r} in the limit where $\mathbf{r} = \mathbf{r}'$. In this case $\hat{\mathbf{J}}(\mathbf{r})$ and $\hat{\mathbf{J}}(\mathbf{r}')$ do not commute and the term is non hermitian. Similar effect arise in the term describing an atom where at some time t a photon interacted, and at time t' another photon interacted. In the limit $t = t'$ the two field components $\hat{\mathbf{D}}(\mathbf{r}, t)$ and $\hat{\mathbf{D}}(\mathbf{r}, t')$ do not commute, and again the term is non hermitian. These two limits both describe possible decay situations of the atomic polarization state, what we need to derive is, that even within the frames of this theory, these decay processes is described by hermitian operators.*

The following section will deal with the observation stated earlier, that we do not know the location of the atoms. A meaningful description of the system must reflect this fact. We already, in equation (5.1.7), mentioned the mean of the density. What we will consider in the following is what to expect when taking the mean of products of densities and spin.

*One might add, especially within the frames of this theory. Lacking ability of this kind could declare failure of the theory.

6.2 Density- and Spin correlations functions

6.2.1 Local Density

Let us begin by considering our definition of the density function (5.1.4).

$$\rho(\mathbf{r}) = \sum_j^{\text{Atoms}} \delta(\mathbf{r} - \mathbf{r}_j) \quad (6.2.1)$$

Let us define the local mean of the atomic density. Doing that we assume that it is possible, at any point in the volume enclosing the atoms, to make a ball enclosing only one atom. This assumption implicitly require that we may neglect atomic collisions. This we argued for on page 28. We define such a ball at a given point, \mathbf{r} , to be $B(\mathbf{r}, R)$. R is the radius of the ball located at position \mathbf{r} . The volume of the ball $B(\mathbf{r}, R)$ we denote by: $|B(\mathbf{r}, R)|$. This way we expect the local mean of the atomic density to be written as:

$$\langle \rho(\mathbf{r}) \rangle_{\text{Mean}} = \frac{1}{|B(\mathbf{r}, R)|} \int_{B(\mathbf{r}, R)} d^3 r' \rho(\mathbf{r}') = \frac{1}{|B(\mathbf{r}, R)|}, \quad (6.2.2)$$

where the integration runs over the volume of the ball $B(\mathbf{r}, R)$.

The position of the atoms we will consider to be a random variable. For some realization of the gas, the j 'th atom may be at some specific location \mathbf{r}_j however at some other realization it may be at a different location. When taking the average over the set of realizations, which means to find the mean density, we need to know the local probability density concerning the realizations. Let us denote this local probability density by $p(\mathbf{r}, \mathbf{r}_j)$, and setting:

$$p(\mathbf{r}_j) = \frac{1}{|B(\mathbf{r}, R)|} \quad \text{then} \quad \int_{B(\mathbf{r}, R)} d^3 r_j p(\mathbf{r}_j) = 1. \quad (6.2.3)$$

We sum over all realizations \mathbf{r}_j . Setting the local probability density as (6.2.3), corresponds to assuming an ideal gas. Each atom is not aware of the presence of any other, and is therefore uncorrelated. Following the theory concerning random functions we find the local average of the random function $\rho(\mathbf{r})$ to be:

$$\langle \rho(\mathbf{r}) \rangle_{\text{Mean}} = \int_{B(\mathbf{r}, R)} d^3 r_j p(\mathbf{r}_j) \rho(\mathbf{r}) = \frac{1}{|B(\mathbf{r}, R)|} \int_{B(\mathbf{r}, R)} d^3 r_j \rho(\mathbf{r}') = \frac{1}{|B(\mathbf{r}, R)|} \quad (6.2.4)$$

as we initially expected.

In order to calculate local density correlations, we need the higher order probability distribu-

tions. Assuming that any probability distribution is uncorrelated, we set the local n-place probability distribution to be simply a product of the respective probability distributions. This means that the chances for finding an atom at one point is independent of having found an atom at another point. Let us calculate the two point correlation function:

$$\langle \rho(\mathbf{r})\rho(\mathbf{r}') \rangle_{\text{Mean}} \quad (6.2.5)$$

The two point local probability density, we denote by:

$$p_2(\mathbf{r}, \mathbf{r}', \mathbf{r}_j) = \frac{1}{|B(\mathbf{r}, R)||B(\mathbf{r}', R')|}. \quad (6.2.6)$$

Depending on the points \mathbf{r} and \mathbf{r}' the balls $B(\mathbf{r}, R)$ and $B(\mathbf{r}', R')$ may intersect, consider Figure 6.3. To shorten the notation, let us define:

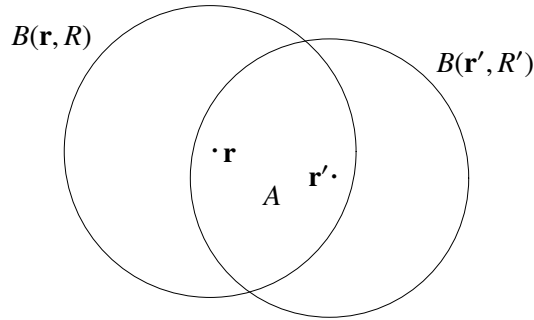


Figure 6.3: The set A is the intersection between the two balls $B(\mathbf{r}, R)$ and $B(\mathbf{r}', R')$.

$$B_1 \equiv B(\mathbf{r}, R) \quad \text{and} \quad B_2 \equiv B(\mathbf{r}', R') \quad (6.2.7)$$

According to our assumption however, the realizations of the random variable \mathbf{r}_j in the set, $A = B_1 \cap B_2$ are the same for both balls. This is best understood, when considering a snapshot of the cloud, corresponding to certain realization of the random variable \mathbf{r}_j . Watching the cloud from a pair of telescopes, one will see two images having nothing in common when focusing the two telescopes at different places in the cloud. If however one focuses at the same spot in the cloud, the image one sees in the two telescopes will always be the same. In that case the realizations of the random variables \mathbf{r}_j are the same for the two images. We therefore have to treat the calculations differently depending on the two locations of the gas we consider. Calculating the two point density correlation function we find:

$$\langle \rho(\mathbf{r})\rho(\mathbf{r}') \rangle_{\text{Mean}} = \int_{B_1} \int_{B_2} p_2(\mathbf{r}, \mathbf{r}', \mathbf{r}_j) \rho(\mathbf{r})\rho(\mathbf{r}') d\mathbf{r}_j. \quad (6.2.8)$$

Where we mean integration over all realizations. Splitting out the integral we may write:

$$\int_{B_1} \int_{B_2} = \int_{B_1 \setminus A} \int_{B_2 \setminus A} + \int_{B_1 \setminus A} \int_A + \int_A \int_{B_2 \setminus A} + \int_A \int_A \quad (6.2.9)$$

The first three terms share no realizations, however the last term will share realizations completely and must be treated differently. Assume now that $\mathbf{r}, \mathbf{r}' \in A$ then

$$\begin{aligned} \langle \rho(\mathbf{r})\rho(\mathbf{r}') \rangle_{\text{Mean}} &= \int_{B_1} \int_{B_2} p_2(\mathbf{r}, \mathbf{r}', \mathbf{r}_j) \rho(\mathbf{r})\rho(\mathbf{r}') d\mathbf{r}_j \\ &= \frac{1}{|B_1||B_2|} \int_A \int_A \sum_j^{\text{Atoms}} \delta(\mathbf{r} - \mathbf{r}_j) \delta(\mathbf{r}' - \mathbf{r}_j) d\mathbf{r}_j \\ &= \frac{1}{|B_1||B_2|} \delta(\mathbf{r} - \mathbf{r}') \int_A d\mathbf{r}_j \\ &= \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \delta(\mathbf{r} - \mathbf{r}') \end{aligned} \quad (6.2.10)$$

Where we used that the term is only contributing when $\mathbf{r} = \mathbf{r}'$ and hence $A = B_1 = B_2$. Considering the real normalized balls we may always for any $\mathbf{r} \neq \mathbf{r}'$ have the two balls to be disjunct. Hence in the case $\mathbf{r} \neq \mathbf{r}'$ only the first term in the integral (6.2.9) contribute, and we may calculate:

$$\begin{aligned} \langle \rho(\mathbf{r})\rho(\mathbf{r}') \rangle_{\text{Mean}} &= \int_{B_1} \int_{B_2} p_2(\mathbf{r}, \mathbf{r}', \mathbf{r}_j) \rho(\mathbf{r})\rho(\mathbf{r}') d\mathbf{r}_j \\ &= \frac{1}{|B_1||B_2|} \int_{B_1} \int_{B_2} \sum_{jj'}^{\text{Atoms}} \delta(\mathbf{r} - \mathbf{r}_j) \delta(\mathbf{r}' - \mathbf{r}'_j) \\ &= \frac{1}{|B_1||B_2|} \\ &= \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \langle \rho(\mathbf{r}') \rangle_{\text{Mean}} \end{aligned} \quad (6.2.11)$$

In this respect the local two point density correlation function may be written as:

$$\langle \rho(\mathbf{r})\rho(\mathbf{r}') \rangle_{\text{Mean}} = \begin{cases} \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \langle \rho(\mathbf{r}') \rangle_{\text{Mean}} & \text{for } \mathbf{r} \neq \mathbf{r}' \\ \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \delta(\mathbf{r} - \mathbf{r}') & \text{for } \mathbf{r} = \mathbf{r}' \end{cases} \quad (6.2.12)$$

In this same respect we will consider the spin of the atoms, $\hat{\mathbf{J}}(\mathbf{r}, t)$. The field describing the spin of the atoms is a quantized vector operator, these properties we will have to conserve in a spatial mean description.

6.2.2 Local Spin Density

Let us consider the spin operator as we defined it in equation (5.1.5).

$$\hat{\mathbf{J}}(\mathbf{r}, t) = \sum_j^{\text{Atoms}} \delta(\mathbf{r} - \mathbf{r}_j) \hat{\mathbf{J}}^j \quad (6.2.13)$$

The variable \mathbf{r}_j we treat as a random variable, and we will keep the assumptions on dealing with an ideal gas. What signifies this spin field from the density of atoms $\rho(\mathbf{r})$ is the spin operator $\hat{\mathbf{J}}^j$. Assuming that there can be no spin correlations between different atoms, we may use the results derived for the local density. We then need to consider the commutative nature of the spin operator. Let us consider the two point spin correlation function:

$$\langle \hat{J}_r(\mathbf{r}, t); \hat{J}_s(\mathbf{r}', t) \rangle_{\text{Mean}} = \int_{B_1} \int_{B_2} p_2(\mathbf{r}, \mathbf{r}', \mathbf{r}_j) \hat{J}_r(\mathbf{r}, t) \hat{J}_s(\mathbf{r}', t) d\mathbf{r}_j \quad (6.2.14)$$

Using the same naive expansion in the integration as in equation (6.2.9), we may consider the two types of integral, either they share realizations completely, or they do not. Considering the first case, we calculate:

$$\begin{aligned} \langle \hat{J}_r(\mathbf{r}, t); \hat{J}_s(\mathbf{r}', t) \rangle_{\text{Mean}} &= \int_{B_1} \int_{B_2} p_2(\mathbf{r}, \mathbf{r}', \mathbf{r}_j) \hat{J}_r(\mathbf{r}, t) \hat{J}_s(\mathbf{r}', t) d\mathbf{r}_j \\ &= \frac{1}{|B_1||B_2|} \int_A \int_A \sum_j^{\text{Atoms}} \delta(\mathbf{r} - \mathbf{r}_j) \delta(\mathbf{r}' - \mathbf{r}_j) \hat{J}_r^j, \hat{J}_s^j d\mathbf{r}_j \\ &= \frac{1}{|B_1||B_2|} \delta(\mathbf{r} - \mathbf{r}') \hat{J}_r^j, \hat{J}_s^j \int_A d\mathbf{r}_j \\ &= \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \delta(\mathbf{r} - \mathbf{r}') \bar{J}_r(\mathbf{r}, t) \bar{J}_s(\mathbf{r}, t). \end{aligned} \quad (6.2.15)$$

We introduced a new notation: $\bar{J}_r(\mathbf{r}, t)$, this spin operator do not really differ from the spin operator $\hat{J}_r^j(t)$ only we changed the countable index j with a continuous index \mathbf{r} meaning we smeared out the spin of an atom over the ball containing it. The commutation relations regarding this spin operator read:

$$[\bar{J}_r(\mathbf{r}, t); \bar{J}_s(\mathbf{r}', t)] = \begin{cases} \varepsilon_{rst} \bar{J}_t(\mathbf{r}, t) & \text{for } \mathbf{r} = \mathbf{r}' \\ 0 & \text{for } \mathbf{r} \neq \mathbf{r}' \end{cases} \quad (6.2.16)$$

where we used definition (2.3.6). The second type of integration, in which case $\mathbf{r} \neq \mathbf{r}'$, returns the following:

$$\begin{aligned}
\langle \hat{J}_r(\mathbf{r}, t); \hat{J}_s(\mathbf{r}', t) \rangle_{\text{Mean}} &= \int_{B_1} \int_{B_2} p_2(\mathbf{r}, \mathbf{r}', \mathbf{r}_j) \hat{J}_r(\mathbf{r}, t) \hat{J}_s(\mathbf{r}', t) d\mathbf{r}_j \\
&= \frac{1}{|B_1||B_2|} \int_A \int_A \sum_{jj'}^{\text{Atoms}} \delta(\mathbf{r} - \mathbf{r}_j) \delta(\mathbf{r}' - \mathbf{r}_{j'}) \hat{J}_r^j, \hat{J}_s^{j'} d\mathbf{r}_j d\mathbf{r}_{j'} \\
&= \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \langle \rho(\mathbf{r}') \rangle_{\text{Mean}} \bar{J}_r(\mathbf{r}, t) \bar{J}_s(\mathbf{r}', t)
\end{aligned} \tag{6.2.17}$$

In equation (6.2.18) we sum up the second order correlation functions for spin of atoms and density of atoms. These equations are necessary when transforming equation (6.1.6) to a mean equation, an equation reflecting the fact that the position of the atoms are completely random.

$$\langle \rho(\mathbf{r}) \rho(\mathbf{r}') \rangle_{\text{Mean}} = \begin{cases} \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \langle \rho(\mathbf{r}') \rangle_{\text{Mean}} & \text{for } \mathbf{r} \neq \mathbf{r}' \\ \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \delta(\mathbf{r} - \mathbf{r}') & \text{for } \mathbf{r} = \mathbf{r}' \end{cases} \tag{6.2.18a}$$

$$\langle \hat{J}_r(\mathbf{r}, t) \hat{J}_s(\mathbf{r}', t) \rangle_{\text{Mean}} = \begin{cases} \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \langle \rho(\mathbf{r}') \rangle_{\text{Mean}} \bar{J}_r(\mathbf{r}, t) \bar{J}_s(\mathbf{r}', t) & \text{for } \mathbf{r} \neq \mathbf{r}' \\ \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \delta(\mathbf{r} - \mathbf{r}') \bar{J}_r(\mathbf{r}, t) \bar{J}_s(\mathbf{r}, t) & \text{for } \mathbf{r} = \mathbf{r}' \end{cases} \tag{6.2.18b}$$

$$\langle \rho(\mathbf{r}) \hat{J}_r(\mathbf{r}', t) \rangle_{\text{Mean}} = \begin{cases} \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \langle \rho(\mathbf{r}') \rangle_{\text{Mean}} \bar{J}_r(\mathbf{r}', t) & \text{for } \mathbf{r} \neq \mathbf{r}' \\ \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \delta(\mathbf{r} - \mathbf{r}') \bar{J}_r(\mathbf{r}, t) & \text{for } \mathbf{r} = \mathbf{r}' \end{cases} . \tag{6.2.18c}$$

When actually making the transformation from individual atoms, to spatial mean, we simply invoke the spatial mean operator, $\langle \cdot \rangle_{\text{Mean}}$ on both sides of the equation (6.1.6). The type of correlation functions that appears are exactly the one displayed in equation (6.2.18).

In order to proceed with the equations we will need to know the Green's functions. Generally this may be a difficult task, and the solution one finds will often depend on the manner of ones search, and the requirements stated by ones needs. From the transformation to mean theory, we

introduced a new set of terms including a deltafunction. Considering one such term:

$$\begin{aligned}
& \left\langle \frac{2i\beta}{\hbar\epsilon_0} \hat{\mathbf{J}}_0(\mathbf{r}) \times \left[\iint_{t_0}^t d^3r' dt' \nabla' \times \nabla' \times \bar{\bar{G}}^+(\mathbf{r}, t | \mathbf{r}', t') \cdot c^2 \beta \frac{1}{2} \lambda(\mathbf{r}') \tilde{\mathbf{D}}_0^+(\mathbf{r}', t') \right] \times \tilde{\mathbf{D}}_0^-(\mathbf{r}, t) \right\rangle_{\text{Mean}} \\
&= \frac{2i\beta^2 c^2}{2\hbar\epsilon_0} \iint_{t_0}^t d^3r' dt' \langle \hat{\mathbf{J}}_0(\mathbf{r}) \lambda(\mathbf{r}') \rangle_{\text{Mean}} \times \left[\nabla' \times \nabla' \times \bar{\bar{G}}^+(\mathbf{r}, t | \mathbf{r}', t') \cdot \tilde{\mathbf{D}}_0^+(\mathbf{r}', t') \right] \times \tilde{\mathbf{D}}_0^-(\mathbf{r}, t) \\
&= \frac{2i\beta^2 c^2}{2\hbar\epsilon_0} \iint_{t_0}^t d^3r' dt' \delta(\mathbf{r} - \mathbf{r}') \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \bar{\mathbf{J}}_0(\mathbf{r}, t) \times \left[\nabla' \times \nabla' \times \bar{\bar{G}}^+(\mathbf{r}, t | \mathbf{r}', t') \cdot \tilde{\mathbf{D}}_0^+(\mathbf{r}', t') \right] \times \tilde{\mathbf{D}}_0^-(\mathbf{r}, t)
\end{aligned} \tag{6.2.19}$$

We used $\lambda(\mathbf{r}) = \rho(\mathbf{r}) - \langle \rho(\mathbf{r}') \rangle_{\text{Mean}}$. We thus need a type of Green's function that will provide us with a meaningful result even in the limit of no spatial propagation of the light field. The other type of Green's function we will consider describes a physical propagation of light fields between points in space. In the following section we derive these proper Green's functions.

6.3 Calculating Green's functions

Let us recall the defining equation for our Green's function, where we consider the Green's function concerning the positively oscillating field component.

$$\left(\left(\frac{\partial}{\partial t} + i\omega_L \right)^2 + c^2 \nabla \times \nabla \times \epsilon^{-1}(\mathbf{r}) \right) \bar{\bar{G}}^+(\mathbf{r}, t | \mathbf{r}_0, t_0) = \bar{\bar{I}} \delta(\mathbf{r} - \mathbf{r}_0) \delta(t - t_0) \tag{6.3.1}$$

We showed in equation (5.3.44), that the Green's function concerning the negatively oscillating field may be found from the positively oscillating by complex conjugation. Let us start these calculations by considering the standard Green's function, describing spatial propagation. The derivation follows standard methods that can be found in, e.g, (42), hence we will not go into great detail.

6.3.1 Green's function for Dipole scatterfield

We consider the equation,

$$\left(\left(\frac{\partial}{\partial t} + i\omega_L \right)^2 - c^2 \nabla^2 \epsilon^{-1}(\mathbf{r}) \right) \bar{\bar{G}}^+(\mathbf{r}, t | \mathbf{r}_0, t_0) = \bar{\bar{I}} \delta(\mathbf{r} - \mathbf{r}_0) \delta(t - t_0). \tag{6.3.2}$$

Our assumption for this part will concern the typical spacial range of the propagator or Green's function. In that respect we assume that $\epsilon(\mathbf{r})$ changes very little over the considered distances, and

we therefore may treat it as constant. Let us also consider the problem in Fourier space, concerning time, in which case equation (6.3.2) may be written as:

$$\left(-(\omega + \omega_L)^2 - \frac{c^2}{\epsilon(\mathbf{r})} \nabla^2\right) \bar{G}^+(\mathbf{r}, \omega | \mathbf{r}_0, \omega_0) = \bar{I} \delta(\mathbf{r} - \mathbf{r}_0). \quad (6.3.3)$$

Assuming the system to be spherical symmetric with no boundary, the Green's function can only depend on $R = |\mathbf{r} - \mathbf{r}_0|$ and from the form of the Laplacian operator in spherical coordinates, \bar{G}_ω^+ has to satisfy:

$$\frac{c^2}{\epsilon(\mathbf{r})} \frac{1}{R} \frac{d^2}{dR^2} R \bar{G}_\omega^+ + (\omega_L + \omega)^2 \bar{G}_\omega^+ = -\delta(\mathbf{R}), \quad (6.3.4)$$

where we used the short hand notation for the fourier transformed Green's function. Solving this differential equation, and correctly normalizing, one may find the following[†]:

$$\bar{G}^+(\mathbf{r}, t | \mathbf{r}_0, t_0) = \bar{I} \delta\left(t - \left\{t_0 + \frac{\sqrt{\epsilon(\mathbf{r})} |\mathbf{r} - \mathbf{r}_0|}{c}\right\}\right) \frac{e^{ik_L |\mathbf{r} - \mathbf{r}_0|}}{|\mathbf{r} - \mathbf{r}_0|} \quad (6.3.5a)$$

$$\bar{G}^-(\mathbf{r}, t | \mathbf{r}_0, t_0) = \bar{I} \delta\left(t - \left\{t_0 + \frac{\sqrt{\epsilon(\mathbf{r})} |\mathbf{r} - \mathbf{r}_0|}{c}\right\}\right) \frac{e^{-ik_L |\mathbf{r} - \mathbf{r}_0|}}{|\mathbf{r} - \mathbf{r}_0|} \quad (6.3.5b)$$

This particular manifestation of the Green's function, simple as it may look, has an obvious disadvantage. It has a singularity for $|\mathbf{r} - \mathbf{r}_0| = 0$. We found that the type of terms we wanted to calculate could be split in cases where $\mathbf{r} \neq \mathbf{r}_0$ and $\mathbf{r} = \mathbf{r}_0$. In the first case the Green's functions (6.3.5) is well suited, however in the second case, we can not rely on standard results. In the following we derive the necessary Green's function using methods similar to ones described in textbooks such as (42; 49; 46; 47)

6.3.2 Green's function for infinitely small propagations

In section 4.2 we described how to quantize the electric displacement $\hat{\mathbf{D}}(\mathbf{r}, t)$ in a polarizable media. We found it extremely helpful to expand the system in a set of basis functions appropriate for this particular system. In the following this set of basisfunctions $\{\mathbf{f}_k(\mathbf{r})\}$, will again prove to simplify the problem considerably. Working in the Gauge $\nabla \phi = 0$ corresponded to considering transverse fields only, and in this space the identity functional can be represented by the distribution: $\bar{\delta}_\tau(\mathbf{r}, \mathbf{r}')$

[†]This particular derivation may be found in (42) pg. 245

Remember how we defined this distribution:

$$\sum_{\mathbf{k}} \frac{\mathbf{f}_{\mathbf{k}}(\mathbf{r})\mathbf{f}_{\mathbf{k}}^*(\mathbf{r}')}{\epsilon(\mathbf{r}')} = \bar{\bar{\delta}}_{\mathbf{T}}(\mathbf{r}, \mathbf{r}') \quad (6.3.6)$$

In this respect the differential equation (6.2.19) may be written:

$$\left(\left(\frac{\partial}{\partial t} + i\omega_{\mathbf{L}} \right)^2 + c^2 \nabla \times \nabla \times \epsilon^{-1}(\mathbf{r}) \right) \bar{G}^+(\mathbf{r}, t | \mathbf{r}_0, t_0) = \bar{\bar{\delta}}_{\mathbf{T}}(\mathbf{r}, \mathbf{r}_0) \delta(t - t_0) \quad (6.3.7)$$

Let us expand the Green's function in the basis $\{\mathbf{f}_{\mathbf{k}}(\mathbf{r})\}$.

$$\bar{G}^+(\mathbf{r}, t | \mathbf{r}_0, t_0) = \sum_{\mathbf{k}} \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \mathbf{A}_{\mathbf{k}}(\mathbf{r}_0, t, t_0) \quad (6.3.8)$$

Using this expansion, along with equation (6.3.6), we may rewrite equation (6.3.7) to the following.

$$\sum_{\mathbf{k}} \left(\left(\frac{\partial}{\partial t} + i\omega_{\mathbf{L}} \right)^2 + c^2 \nabla \times \nabla \times \epsilon^{-1}(\mathbf{r}) \right) \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \mathbf{A}_{\mathbf{k}}(\mathbf{r}_0, t, t_0) = \sum_{\mathbf{k}} \frac{\mathbf{f}_{\mathbf{k}}(\mathbf{r})\mathbf{f}_{\mathbf{k}}^*(\mathbf{r}')}{\epsilon(\mathbf{r}')} \delta(t - t_0) \quad (6.3.9)$$

From the way we defined the basisfunctions $\{\mathbf{f}_{\mathbf{k}}(\mathbf{r})\}$, equation (4.2.9), we may let the differential operator $\nabla \times \nabla \times \epsilon^{-1}(\mathbf{r})$ work on the function $\mathbf{f}_{\mathbf{k}}(\mathbf{r})$ to get the simple equation:

$$\sum_{\mathbf{k}} \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \left(\left(\frac{\partial}{\partial t} + i\omega_{\mathbf{L}} \right)^2 + \omega_{\mathbf{k}}^2 \right) \mathbf{A}_{\mathbf{k}}(\mathbf{r}_0, t, t_0) = \sum_{\mathbf{k}} \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \delta(t - t_0) \frac{\mathbf{f}_{\mathbf{k}}^*(\mathbf{r}')}{\epsilon(\mathbf{r}')} \quad (6.3.10)$$

From this equation we see that the expansions coefficients $\mathbf{A}_{\mathbf{k}}(\mathbf{r}_0, t, t_0)$ must be expressed as some function of time times the \mathbf{k} -th basisfunction divided by $\epsilon(\mathbf{r}')$:

$$\mathbf{A}_{\mathbf{k}}(\mathbf{r}_0, t, t_0) = g_{\mathbf{k}}(t, t_0) \frac{\mathbf{f}_{\mathbf{k}}^*(\mathbf{r}')}{\epsilon(\mathbf{r}')} \quad (6.3.11)$$

The function $g_{\mathbf{k}}(t, t_0)$ has to fulfill the much simpler time differential equation:

$$\left(\left(\frac{\partial}{\partial t} + i\omega_{\mathbf{L}} \right)^2 + \omega_{\mathbf{k}}^2 \right) g_{\mathbf{k}}(t, t_0) = \delta(t - t_0) \quad (6.3.12)$$

Similar calculations regarding the reciprocal equation discussed on page 59 gives us the other useful time differential equation.

$$\left(\left(\frac{\partial}{\partial t_0} - i\omega_{\mathbf{L}} \right)^2 + \omega_{\mathbf{k}}^2 \right) g_{\mathbf{k}}(t, t_0) = \delta(t - t_0) \quad (6.3.13)$$

First we consider the simple case where $t \neq t_0$. From the two equations (6.3.12) and (6.3.13) we derive a general solution for the function $g_{\mathbf{k}}(t, t_0)$.

$$g_{\mathbf{k}}(t, t_0) = \begin{cases} \sqrt{\frac{1}{2}}(A(t_0)e^{i\omega_{\mathbf{k}}t} + B(t_0)e^{-i\omega_{\mathbf{k}}t})e^{i\omega_{\mathbf{L}}t} & , t < t_0 \\ \sqrt{\frac{1}{2}}(A'(t_0)e^{i\omega_{\mathbf{k}}t} + B'(t_0)e^{-i\omega_{\mathbf{k}}t})e^{i\omega_{\mathbf{L}}t} & , t > t_0 \\ \sqrt{\frac{1}{2}}(C(t)e^{i\omega_{\mathbf{k}}t_0} + D(t)e^{-i\omega_{\mathbf{k}}t_0})e^{-i\omega_{\mathbf{L}}t} & , t < t_0 \\ \sqrt{\frac{1}{2}}(C'(t)e^{i\omega_{\mathbf{k}}t_0} + D'(t)e^{-i\omega_{\mathbf{k}}t_0})e^{-i\omega_{\mathbf{L}}t} & , t > t_0 \end{cases}, \quad (6.3.14)$$

where we included a normalization factor, and the unknown functions $A(t), \dots$ are all normalized accordingly. If we choose to look at the retarded Green's function only, we may reduce equation (6.3.14) to the shorter form:

$$g_{\mathbf{k}}(t, t_0) = \begin{cases} \frac{1}{2}C e^{i\omega_{\mathbf{k}}(t-t_0)} e^{i\omega_{\mathbf{L}}(t-t_0)} & t > t_0 \\ \frac{1}{2}C e^{-i\omega_{\mathbf{k}}(t-t_0)} e^{i\omega_{\mathbf{L}}(t-t_0)} & t < t_0 \end{cases} \quad (6.3.15)$$

For the retarded Green's function we will only have to find the constant C . To do that we use the information we have on $g_{\mathbf{k}}(t, t_0)$ in the case where $t = t_0$. Equation (6.3.12) may be written in the form:

$$\delta(t - t_0) = \frac{\partial^2}{\partial t^2} g_{\mathbf{k}}(t, t_0) + 2i\omega_{\mathbf{L}} \frac{\partial}{\partial t} g_{\mathbf{k}}(t, t_0) - \omega_{\mathbf{L}}^2 g_{\mathbf{k}}(t, t_0) - \omega_{\mathbf{k}}^2 g_{\mathbf{k}}(t, t_0)$$

This equation we integrate up over t from $t = t_0 - \epsilon$ to $t = t_0 + \epsilon$, and find,

$$1 = \left[\frac{\partial}{\partial t} g_{\mathbf{k}}(t, t_0) \right]_{t_0-\epsilon}^{t_0+\epsilon} + 2i\omega_{\mathbf{L}} \left[g_{\mathbf{k}}(t, t_0) \right]_{t_0-\epsilon}^{t_0+\epsilon} - \omega_{\mathbf{L}}^2 \int_{t_0-\epsilon}^{t_0+\epsilon} dt' g_{\mathbf{k}}(t', t_0) - \omega_{\mathbf{k}}^2 \int_{t_0-\epsilon}^{t_0+\epsilon} g_{\mathbf{k}}(t, t_0)$$

Letting $\epsilon \rightarrow 0$ only one term is non vanishing, and we are left with:

$$1 = \left\{ \frac{\partial}{\partial t} g_{\mathbf{k}}(t, t_0) \right\}_{t_0+\epsilon} - \left\{ \frac{\partial}{\partial t} g_{\mathbf{k}}(t, t_0) \right\}_{t_0-\epsilon} \quad (6.3.16)$$

This equation we may readily solve using equation (6.3.15), and the two terms give the following.

$$\begin{aligned} \left\{ \frac{\partial}{\partial t} g_{\mathbf{k}}(t, t_0) \right\}_{t_0+\epsilon} &= C \frac{1}{2} (i\omega_{\mathbf{k}} + i\omega_{\mathbf{L}}) e^{i\omega_{\mathbf{k}}\epsilon} e^{i\omega_{\mathbf{L}}\epsilon} \\ \left\{ \frac{\partial}{\partial t} g_{\mathbf{k}}(t, t_0) \right\}_{t_0-\epsilon} &= C \frac{1}{2} (-i\omega_{\mathbf{k}} + i\omega_{\mathbf{L}}) e^{-i\omega_{\mathbf{k}}\epsilon} e^{i\omega_{\mathbf{L}}\epsilon} \end{aligned} \quad (6.3.17)$$

Inserting this result into equation (6.3.16) we find the constant C to be $C = -i/\omega_{\mathbf{k}}$. Considering only the case $t > t_0$, that is propagating forward in time, the Green's function $\bar{G}^{\pm}(\mathbf{r}, t|\mathbf{r}_0, t_0)$ may

be written in the form:

$$\bar{\bar{G}}^+(\mathbf{r}, t|\mathbf{r}_0, t_0) = -i \sum_{\mathbf{k}} \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \mathbf{f}_{\mathbf{k}}^*(\mathbf{r}_0) \frac{e^{i(\omega_{\mathbf{k}} - \omega_L)(t-t_0)}}{\omega_{\mathbf{k}} \epsilon(\mathbf{r})} \quad (6.3.18a)$$

$$\bar{\bar{G}}^-(\mathbf{r}, t|\mathbf{r}_0, t_0) = i \sum_{\mathbf{k}} \mathbf{f}_{\mathbf{k}}^*(\mathbf{r}) \mathbf{f}_{\mathbf{k}}(\mathbf{r}_0) \frac{e^{-i(\omega_{\mathbf{k}} - \omega_L)(t-t_0)}}{\omega_{\mathbf{k}} \epsilon(\mathbf{r})} \quad (6.3.18b)$$

This set of Green's functions (6.3.5) and (6.3.18a), in principle solves our problem. We only have to implement this set of Green's functions into equation (6.1.6), and the rest is merely calculations. In the following section we will deal with these apparently non hermitian terms in equation (6.1.6) discussed on page 72, and we will show that using our Green's functions the equation do turn out to be hermitian.

6.4 Lowest order solution of equation of motion for atoms and fields

In the equation (6.1.6), we consider interaction processes to second order. The first order term we will deal with, when considering the corresponding single mode equation in chapter 7. In respect to the discussion regarding hermitian terms, we will, by way of introduction, show that the first order term is in fact hermitian. A simple calculation reveals:

$$\left(\mathbf{A} \times \mathbf{B}\right)_s^\dagger = \varepsilon_{snm} \left(A_n B_m\right)^\dagger = \varepsilon_{snm} B_m^\dagger A_n^\dagger = -\left(\mathbf{B}^\dagger \times \mathbf{A}^\dagger\right). \quad (6.4.1)$$

Using this equation, we may show that the hermitian conjugate of the first order term of equation (6.1.6), in the mean picture, returns:

$$\left(\langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \frac{2i\beta}{\hbar \epsilon_0} \hat{\mathbf{J}}_0(\mathbf{r}) \times \tilde{\mathbf{D}}_0^+(\mathbf{r}, t) \times \tilde{\mathbf{D}}_0^-(\mathbf{r}, t)\right)^\dagger = \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \frac{2i\beta}{\hbar \epsilon_0} \hat{\mathbf{J}}_0(\mathbf{r}) \times \tilde{\mathbf{D}}_0^+(\mathbf{r}, t) \times \tilde{\mathbf{D}}_0^-(\mathbf{r}, t) \quad (6.4.2)$$

Calmed by this calculation we will continue with the some what more troublesome terms of second order in β .

6.4.1 Examining the Second order Solution

The simplest term we may consider is the one given in equation (6.2.19) we write it diagrammatically as shown in Figure 6.4. What is interesting for us to know in order to calculate this integral is the meaning of the expression $\nabla' \times \nabla' \times \bar{\bar{G}}^+(\mathbf{r}, t|\mathbf{r}', t')$. Using the infinitely small propagation Green's function derived in section 6.3.2, along with the definition of the basis functions $\{\mathbf{f}_{\mathbf{k}}(\mathbf{r})\}$

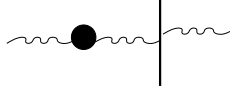


Figure 6.4: Diagrammatic representation of the term (6.2.19)

given in equation (4.2.9), we find in the limit where we treat $\epsilon(\mathbf{r})$ constant, the following:

$$\nabla' \times \nabla' \times \bar{G}^+(\mathbf{r}, t | \mathbf{r}', t') = -i \sum_{\mathbf{k}} \frac{\omega_{\mathbf{k}}}{c^2} \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \mathbf{f}_{\mathbf{k}}^*(\mathbf{r}_0) e^{i(\omega_{\mathbf{k}} - \omega_L)(t - t_0)} \quad (6.4.3)$$

We then use the same expansions of the basisfunctions $\{\mathbf{f}_{\mathbf{k}}(\mathbf{r})\}$ as done in section 4.2.4 equation (4.2.48). The general integral given with solution in equation (6.4.4), where $\mathbf{n}(\mathbf{r}', t)$ is some vectorfield, one may readily calculate. The calculations are very similar to the calculations made in section 4.2.4, one considers the problem in polar coordinates, and by clever choice of polarizations vectors the integrations simplify.

$$\int d^3 r \delta(\mathbf{r} - \mathbf{r}') \nabla' \times \nabla' \times \bar{G}^+(\mathbf{r}, t | \mathbf{r}', t') \cdot \mathbf{n}(\mathbf{r}', t') = \frac{-i k_L^3 \epsilon(\mathbf{r})}{c^2 3\pi} \delta(t - t') e^{i\omega_L(t - t')} \mathbf{n}(\mathbf{r}, t') \quad (6.4.4)$$

The term depicted in Figure 6.4, reduce to the following:

$$\langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \frac{\beta^2 k_L^3 \epsilon(\mathbf{r})}{\hbar \epsilon_0 3\pi} \bar{\mathbf{J}}_0(\mathbf{r}) \times (\tilde{\mathbf{D}}_0^+(\mathbf{r}, t) \times \tilde{\mathbf{D}}_0^-(\mathbf{r}, t)), \quad (6.4.5)$$

where we also approximated $\omega_{\mathbf{k}} \approx \omega_L$ whenever this factor did not occur in an exponential. The similar term, where it is the negatively oscillating part of the displaced electric field, $\tilde{\mathbf{D}}_0^-(\mathbf{r}, t)$ that first bounced of an atom, gives a similar result, however the sign changed accordingly, since $\bar{G}^- = (\bar{G}^+)^*$, and we get the following:

$$-\langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \frac{\beta^2 k_L^3 \epsilon(\mathbf{r})}{\hbar \epsilon_0 3\pi} \bar{\mathbf{J}}_0(\mathbf{r}) \times (\tilde{\mathbf{D}}_0^+(\mathbf{r}, t) \times \tilde{\mathbf{D}}_0^-(\mathbf{r}, t)) \quad (6.4.6)$$

Fortunately these two terms cancels, (6.4.5) and (6.4.6). What one notices when calculating the hermitian of each term, is that both terms are anti hermitian. This means that each term reveals an unphysical phase, initially excluded by Gauge invariance.[‡] These terms fortunately cancel thus our physical predictions being, in this respect, consistent.

The next type of terms left to consider is the terms describing a photon interacting first with one atom and then secondly with another atom. The terms is depicted in Figure 6.5 Using a shorthand notation suppressing all time and space dependence, and integrations, the term may be

[‡]See, e.g., (4) pg. 146.

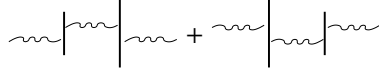


Figure 6.5: Diagrammatic representation of two interaction processes.

represented as:

$$-\frac{2\beta^2 c^2}{\hbar\epsilon_0} \hat{\mathbf{J}} \times \left\{ \left(\bar{\mathbf{P}}^+ \cdot [\hat{\mathbf{J}}' \times \tilde{\mathbf{D}}'^+] \right) \times \tilde{\mathbf{D}}^- - \tilde{\mathbf{D}}^+ \times \left(\bar{\mathbf{P}}^- \cdot [\hat{\mathbf{J}}' \times \tilde{\mathbf{D}}'^-] \right) \right\} \quad (6.4.7)$$

where we introduced the wavepropagator $\bar{\mathbf{P}}^+ = \nabla' \times \nabla' \times \bar{\mathbf{G}}^+$. When taking the spatial mean over positions of atoms, we split the term in two types, one of them including a deltafunction. The term without the delta function is an integral in which $\mathbf{r} \neq \mathbf{r}'$, and hence the spin operators always commute. We may check if this term is hermitian using the property on commutation of spin, along with equation (6.4.1). Simple calculation shows that this term is in fact hermitian, and we conclude, that in the case where $\mathbf{r} \neq \mathbf{r}'$, and hence the involved spin operators commute, the term in its current form is physically meaningful.

In the term including the deltafunction the integration leaves the two spin operators to be non-commuting, and a combination of measuring first the x -component of the spin, and secondly the y -component, may effectively correspond to measuring the z -component times an additional factor, as we will show. Since the two spin operators do not commute, we cannot know if this term is hermitian. In fact calculations will show that the term is not hermitian.

Using the expansion of the Green's function given in equation (6.4.3), and otherwise proceed as done with integral (6.4.4), one may find the term including the spatial deltafunction to reduce to:

$$\langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \frac{2i\beta^2 k_L^3 \epsilon(\mathbf{r})}{\hbar\epsilon_0} \bar{\mathbf{J}} \times \left\{ [\bar{\mathbf{J}} \times \tilde{\mathbf{D}}^+] \times \tilde{\mathbf{D}}^- + \tilde{\mathbf{D}}^+ [\bar{\mathbf{J}} \times \tilde{\mathbf{D}}^-] \right\}. \quad (6.4.8)$$

Again we suppress space and time dependence. Still we need to figure out what this term means. The before mentioned effect of first letting one component of the spin operator work, and second another, will still have to be sorted out. Simple calculations using the definition of the spin operator (2.2.19) shows, e.g.

$$\bar{J}_z \bar{J}_x = \frac{1}{4} (|2\rangle\langle 2| - |1\rangle\langle 1|)(|2\rangle\langle 1| - |1\rangle\langle 2|) = \frac{1}{4} (|2\rangle\langle 1| - |1\rangle\langle 2|) = \frac{i}{2} \bar{J}_y. \quad (6.4.9)$$

Similar calculations will give the remaining products. These spin products can be put in the short

form:

$$\bar{J}_n \bar{J}_m = \epsilon_{nml} \frac{i}{2} \bar{J}_l \quad (6.4.10a)$$

$$\bar{J}_x^2 = \bar{J}_y^2 = \bar{J}_z^2 = \frac{1}{4} \quad (6.4.10b)$$

Consider the term (6.4.8). We calculate this, by putting it on component form, and using equation (6.4.10). This way it reduces to the following, written on vector component form. A detailed calculation can be found in Appendix D.1

$$\langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \frac{2\beta^2 k_L^3 \epsilon(\mathbf{r})}{\hbar \epsilon_0 3\pi} \left(\begin{array}{l} \frac{1}{2} \bar{J}_z (\tilde{D}_z^+ \tilde{D}_x^- - \tilde{D}_x^+ \tilde{D}_z^-) + \frac{1}{2} \bar{J}_y (\tilde{D}_y^+ \tilde{D}_x^- - \tilde{D}_x^+ \tilde{D}_y^-) + \frac{i}{2} (\tilde{D}_z^+ \tilde{D}_y^- - \tilde{D}_y^+ \tilde{D}_z^-) \\ \frac{1}{2} \bar{J}_x (\tilde{D}_x^+ \tilde{D}_y^- - \tilde{D}_y^+ \tilde{D}_x^-) + \frac{1}{2} \bar{J}_z (\tilde{D}_z^+ \tilde{D}_y^- - \tilde{D}_y^+ \tilde{D}_z^-) + \frac{i}{2} (\tilde{D}_x^+ \tilde{D}_z^- - \tilde{D}_z^+ \tilde{D}_x^-) \\ \frac{1}{2} \bar{J}_y (\tilde{D}_x^+ \tilde{D}_y^- - \tilde{D}_y^+ \tilde{D}_x^-) + \frac{1}{2} \bar{J}_x (\tilde{D}_x^+ \tilde{D}_z^- - \tilde{D}_z^+ \tilde{D}_x^-) + \frac{i}{2} (\tilde{D}_y^+ \tilde{D}_x^- - \tilde{D}_x^+ \tilde{D}_y^-) \end{array} \right) \quad (6.4.11)$$

This term is, as already unveiled, not hermitian, in fact two thirds of this term is anti hermitian. To solve this problem, we turn to the last remaining term, depicted in Figure 6.6 The term is in all

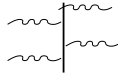


Figure 6.6: Diagrammatic interpretation of two different photons interacting with the same atom.

its glory given in the following form:

$$-\langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \frac{4\beta^2}{\hbar^2 \epsilon_0^2} \left[\int_{t_0}^t dt' \bar{\mathbf{J}}_0(\mathbf{r}) \times \tilde{\mathbf{D}}_0^+(\mathbf{r}, t') \times \tilde{\mathbf{D}}_0^-(\mathbf{r}, t') \right] \times \tilde{\mathbf{D}}_0^+(\mathbf{r}, t) \times \tilde{\mathbf{D}}_0^-(\mathbf{r}, t). \quad (6.4.12)$$

In section 4.2.4 page 46 we noted the equal space commutation relations. These are not trivial, and especially at equal times, they are important. Keeping this in mind, we see that the term (6.4.12) is not hermitian at every point in time. In fact when $t = t'$ interesting things happens, and we will show that terms are being generated. Terms that will make the anti hermitian part of equation (6.4.11) hermitian!

Our line of approach on this task will be to bring equation (6.4.12) on ordered form. To bring it on ordered form means to make sure that for any physical field configuration, we measure only photons when present. Written in terms of creation and annihilation operators, such an ordered

form would be:

$$\underbrace{\hat{a}^\dagger \dots \hat{a}^\dagger}_n \underbrace{\hat{a} \dots \hat{a}}_n \quad (6.4.13)$$

averaging the photon density in vacuum $|0\rangle$ we get zero as expected.

$$\langle 0 | \underbrace{\hat{a}^\dagger \dots \hat{a}^\dagger}_n \underbrace{\hat{a} \dots \hat{a}}_n | 0 \rangle = 0. \quad (6.4.14)$$

Similarly we bring equation (6.4.12) on ordered form. When writing the equation on vector component form and making the cross product, many different terms will appear, consider the type of term:

$$\tilde{D}_x^+(t') \tilde{D}_y^-(t') \tilde{D}_x^+(t) \tilde{D}_y^-(t).$$

Since the two field components $\tilde{D}_y^-(t')$ and $\tilde{D}_x^+(t)$ commute, this term is easily brought to ordered form.

$$\tilde{D}_x^+(t') \tilde{D}_x^+(t) \tilde{D}_y^-(t') \tilde{D}_y^-(t)$$

However other terms such as:

$$\tilde{D}_x^+(t') \tilde{D}_y^-(t') \tilde{D}_y^+(t) \tilde{D}_x^-(t)$$

prove less trivial. When bringing this term to ordered form, we use commutation relation (4.2.51c), and find the following:

$$\begin{aligned} & \tilde{D}_x^+(t') \tilde{D}_y^+(t) \tilde{D}_y^-(t') \tilde{D}_x^-(t) + \tilde{D}_x^+(t') [\tilde{D}_y^-(t) \tilde{D}_y^+(t')] \tilde{D}_x^-(t) \\ &= \tilde{D}_x^+(t') \tilde{D}_y^+(t) \tilde{D}_y^-(t') \tilde{D}_x^-(t) + \frac{\hbar k_L^3 \epsilon(\mathbf{r}) \epsilon_0}{6\pi} \tilde{D}_x^+(t') \tilde{D}_x^-(t) \delta(t - t') \end{aligned} \quad (6.4.15)$$

Thus bringing equation (6.4.12) on ordered form, introduces extra terms. When carefully writing out all vector components, and by using commutation relation (4.2.51c), the thereby acquired terms

read:

$$\begin{aligned}
& - \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \frac{4\beta^2}{\hbar^2 \epsilon_0^2} \int_{t_0}^t dt' \frac{\hbar k_L^3 \epsilon_0 \epsilon(\mathbf{r})}{6\pi} \delta(t-t') \left\{ \begin{array}{l} \left(\begin{array}{l} \bar{J}_y \tilde{D}_y^+(t') \tilde{D}_x^-(t) + \bar{J}_z \tilde{D}_z^+(t') \tilde{D}_x^-(t) \\ \bar{J}_x \tilde{D}_x^+(t') \tilde{D}_y^-(t) + \bar{J}_z \tilde{D}_z^+(t') \tilde{D}_y^-(t) \\ \bar{J}_x \tilde{D}_x^+(t') \tilde{D}_z^-(t) + \bar{J}_y \tilde{D}_y^+(t') \tilde{D}_z^-(t) \end{array} \right) \\ + \left(\begin{array}{l} \bar{J}_x (2\tilde{D}_x^+(t') \tilde{D}_x^-(t) + \tilde{D}_y^+(t') \tilde{D}_y^-(t) + \tilde{D}_z^+(t') \tilde{D}_z^-(t)) \\ \bar{J}_y (\tilde{D}_x^+(t') \tilde{D}_x^-(t) + 2\tilde{D}_y^+(t') \tilde{D}_y^-(t) + \tilde{D}_z^+(t') \tilde{D}_z^-(t)) \\ \bar{J}_z (\tilde{D}_x^+(t') \tilde{D}_x^-(t) + \tilde{D}_y^+(t') \tilde{D}_y^-(t) + 2\tilde{D}_z^+(t') \tilde{D}_z^-(t)) \end{array} \right) \end{array} \right\} \quad (6.4.16)
\end{aligned}$$

Doing the time integration, the terms reduce to a form directly comparable to equation (6.4.11). More importantly we see that some of the terms in equation (6.4.11) combine with terms in equation (6.4.16), and the result, as we will show, turn out to be hermitian.

$$\begin{aligned}
& \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \frac{2\beta^2}{\hbar \epsilon_0} \frac{k_L^3 \epsilon(\mathbf{r})}{3\pi} \left(\begin{array}{l} -\bar{J}_z \frac{1}{2} (\tilde{D}_z^+ \tilde{D}_x^- + \tilde{D}_x^+ \tilde{D}_z^-) - \bar{J}_y \frac{1}{2} (\tilde{D}_y^+ \tilde{D}_x^- + \tilde{D}_x^+ \tilde{D}_y^-) \\ -\bar{J}_x \frac{1}{2} (\tilde{D}_x^+ \tilde{D}_y^- + \tilde{D}_y^+ \tilde{D}_x^-) - \bar{J}_z \frac{1}{2} (\tilde{D}_z^+ \tilde{D}_y^- + \tilde{D}_y^+ \tilde{D}_z^-) \\ -\bar{J}_y \frac{1}{2} (\tilde{D}_y^+ \tilde{D}_z^- + \tilde{D}_z^+ \tilde{D}_y^-) - \bar{J}_x \frac{1}{2} (\tilde{D}_x^+ \tilde{D}_z^- + \tilde{D}_z^+ \tilde{D}_x^-) \end{array} \right) \\
& + \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \frac{2\beta^2}{\hbar \epsilon_0} \frac{k_L^3 \epsilon(\mathbf{r})}{3\pi} \left(\begin{array}{l} \frac{i}{2} (\tilde{D}_z^+ \tilde{D}_y^- - \tilde{D}_y^+ \tilde{D}_z^-) \\ \frac{i}{2} (\tilde{D}_x^+ \tilde{D}_z^- - \tilde{D}_z^+ \tilde{D}_x^-) \\ \frac{i}{2} (\tilde{D}_y^+ \tilde{D}_x^- - \tilde{D}_x^+ \tilde{D}_y^-) \end{array} \right) \\
& - \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \frac{2\beta^2}{\hbar \epsilon_0} \frac{k_L^3 \epsilon(\mathbf{r})}{3\pi} \left(\begin{array}{l} \bar{J}_x (2\tilde{D}_x^+(t') \tilde{D}_x^-(t) + \tilde{D}_y^+(t') \tilde{D}_y^-(t) + \tilde{D}_z^+(t') \tilde{D}_z^-(t)) \\ \bar{J}_y (\tilde{D}_x^+(t') \tilde{D}_x^-(t) + 2\tilde{D}_y^+(t') \tilde{D}_y^-(t) + \tilde{D}_z^+(t') \tilde{D}_z^-(t)) \\ \bar{J}_z (\tilde{D}_x^+(t') \tilde{D}_x^-(t) + \tilde{D}_y^+(t') \tilde{D}_y^-(t) + 2\tilde{D}_z^+(t') \tilde{D}_z^-(t)) \end{array} \right) \quad (6.4.17)
\end{aligned}$$

Showing that the terms in equation (6.4.17) are hermitian is not difficult considering the calculation below.

$$\left[\bar{J}_z \frac{1}{2} (\tilde{D}_z^+ \tilde{D}_x^- + \tilde{D}_x^+ \tilde{D}_z^-) \right]^\dagger = \bar{J}_z \frac{1}{2} (\tilde{D}_x^+ \tilde{D}_z^- + \tilde{D}_z^+ \tilde{D}_x^-) \quad (6.4.18)$$

Similarly the remaining terms may be checked. In principle we also have to check that the remaining ordered terms, are in fact hermitian. It is merely a matter of writing out the field components, making all the cross products, and then checking, using same procedure as in equation (6.4.18), that the terms are hermitian. In Appendix D.2 we have written out the complete set of ordered terms. We conclude this chapter by reminding ourselves that we are now dealing with an equation describing the spin of the atoms, that turned out to be hermitian. In the next chapter we will reduce the equation, by considering a single mode light field. We will also consider the remaining integral from equation (6.4.7) where $\mathbf{r} \neq \mathbf{r}'$.

Single mode solution for the spin-equation

In this chapter we derive the main result for this thesis work. We have at our hand, an equation telling us to second order in β how the evolution of the atomic spin is going to be when perturbed by an off-resonant laser field. The approximation we, at this point, still have not used is the spatial single mode nature of an incoming laser field. Generally if we wanted to predict an evolution of a specific system, we would have to take the quantum mechanical mean of the physical operator in consideration, for a given set of initial conditions. That means the expectation value of, e.g., the spin is given by:

$$\langle \hat{\mathbf{J}} \rangle = \langle \psi_{\text{init}} | \hat{\mathbf{J}} | \psi_{\text{init}} \rangle \quad (7.1.1)$$

where ψ_{init} is the quantum mechanical initial state. In our case the initial state would be an outer product of the initial spin state ψ_{init}^S , and initial field state ψ_{init}^F . Assuming that we are only going to let the initial field state be expanded in a single spatial mode \mathbf{k} , we know from the ordered nature of equation (6.1.6), that no terms other than the ones dealing with this particular mode \mathbf{k} are going to contribute anything. This of course follows from the nature of the annihilation operator working on a vacuum state.

$$\hat{a}|0_{\mathbf{k}}\rangle = 0 \quad (7.1.2)$$

In the following we will consider an initial field state propagation in the direction denoted by the z -coordinate.

7.2 Single mode representation of equation (6.1.6)

In most physical applications we are dealing with a propagation light pulse. This property of the field we would like to appear explicitly in our equations. In the simplest case we could consider a propagating coherent field.* A pulse is then easily described by letting the eigenvalue of the coherent state, with respect to the annihilation operator, depend on position and time.

$$\hat{a}_{\mathbf{k}}|\alpha_{\mathbf{k}}\rangle = \alpha_{\mathbf{k}}(\mathbf{r}, t)|\alpha_{\mathbf{k}}\rangle \quad (7.2.1)$$

This eigenvalue contains information on both the intensity of the light field, but also on evolution of the pulse. Splitting the eigenvalue in a normalized part describing the evolution of the light pulse $u_{\mathbf{k}}(\mathbf{r}, t)$, and a part describing the intensity of the light field $\alpha_{\mathbf{k}}$. We see that we might as well include the scalerfield describing the evolution of the light pulse, in the general expansion of the displaced electric field $\hat{\mathbf{D}}(\mathbf{r}, t)$ equation (4.2.39), and then treat the coherent state in the usual way. In this respect we write the single mode displaced electric field in the following way:

$$\hat{\mathbf{D}}_{\mathbf{k}}(\mathbf{r}, t) = \tilde{\mathbf{D}}_{\mathbf{k}}^+(\mathbf{r}, t)e^{i\omega_{\mathbf{L}}t} + \tilde{\mathbf{D}}_{\mathbf{k}}^-(\mathbf{r}, t)e^{-i\omega_{\mathbf{L}}t}, \quad (7.2.2a)$$

where

$$\hat{\mathbf{D}}_{\mathbf{k}}^+(\mathbf{r}, t) = -i\sqrt{\frac{\hbar\omega_{\mathbf{k}}\epsilon_0}{2}}u_{\mathbf{k}}^*(\mathbf{r}, t)\hat{a}_{\mathbf{k}}^{\dagger}e^{i(\omega_{\mathbf{k}}-\omega_{\mathbf{L}})t}\mathbf{f}_{\mathbf{k}}^*(\mathbf{r}) \quad (7.2.2b)$$

$$\hat{\mathbf{D}}_{\mathbf{k}}^-(\mathbf{r}, t) = i\sqrt{\frac{\hbar\omega_{\mathbf{k}}\epsilon_0}{2}}u_{\mathbf{k}}(\mathbf{r}, t)\hat{a}_{\mathbf{k}}e^{-i(\omega_{\mathbf{k}}-\omega_{\mathbf{L}})t}\mathbf{f}_{\mathbf{k}}(\mathbf{r}) \quad (7.2.2c)$$

The slowly varying pulse function $u_{\mathbf{k}}(\mathbf{r}, t)$ is normalized such that $\int d^3r|u_{\mathbf{k}}(\mathbf{r}, t)|^2 = 1$ In the following section we will consider the first order term of the spatial mean version of equation (6.1.6).

7.2.1 Considering the First order Solution

The first order term in consideration is given generally by the following expression.

$$\langle\rho(\mathbf{r})\rangle_{\text{Mean}}\frac{2i\beta}{\hbar\epsilon_0}\mathbf{J}_0(\mathbf{r})\times\tilde{\mathbf{D}}_0^+(\mathbf{r}, t)\times\tilde{\mathbf{D}}_0^-(\mathbf{r}, t) \quad (7.2.3)$$

The diagrammatic representation of the first order term is depicted in Figure 7.1. We will here and in the following choose the single mode such that we may put $\tilde{D}_z(\mathbf{r}, t) = 0$. This introduces a little

*For discussion of coherent fields check, e.g., (36)

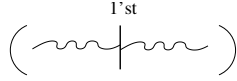


Figure 7.1: Diagrammatic representation of the first order perturbative term.

different way to define the coordinate system used. In order to keep $\tilde{D}_z(\mathbf{r}, t) = 0$ a meaningful assumption, we must choose a local coordinate system in which we always have $\mathbf{e}_z \cdot \mathbf{k}(\mathbf{r}) = 0$. This essentially means we work in a basis as defined in Figure 4.1. Doing this we may write equation (7.2.3) on the following form.

$$-\frac{2\langle\rho(\mathbf{r})\rangle_{\text{Mean}}\beta\omega_L\epsilon(\mathbf{r})|u_{\mathbf{k}}(\mathbf{r}, t)|^2}{V}\hat{S}_z\begin{pmatrix} \bar{J}_y \\ -\bar{J}_x \\ 0 \end{pmatrix} \quad (7.2.4)$$

We introduced local Stokes operators similar to the ones defined in equation (2.3.17). We used a local version of the basisfunctions $\mathbf{f}_{\mathbf{k}}(\mathbf{r}, t)$ similar to equation (4.2.48). In this case however we normalized according to a quantization box of volume V . A detailed calculation leading to equation (7.2.4) can be found in Appendix D.3. This equation is, not surprisingly, similar to equation (2.3.20a). Changing to global Stokes operators, we would find exactly equation (2.3.20a), though with an additional factor of $\epsilon(\mathbf{r})$. A factor we already sought in the discussion, section 2.5 page 24. Let us now consider the higher order terms. First we consider the decay-like terms given in equation (6.4.17).

7.2.2 Second order Solutions: Dealing with decay

Setting $\tilde{D}_z(\mathbf{r}, t) = 0$ in equation (6.4.17), leaves us with the following set of terms.

$$\langle\rho(\mathbf{r})\rangle_{\text{Mean}}\frac{2\beta^2 k_L^3\epsilon(\mathbf{r})}{\hbar\epsilon_0 3\pi}\left\{\begin{pmatrix} -\bar{J}_y\frac{1}{2}(\tilde{D}_y^+\tilde{D}_x^- + \tilde{D}_x^+\tilde{D}_y^-) \\ -\bar{J}_x\frac{1}{2}(\tilde{D}_x^+\tilde{D}_y^- + \tilde{D}_y^+\tilde{D}_x^-) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{i}{2}(\tilde{D}_y^+\tilde{D}_x^- - \tilde{D}_x^+\tilde{D}_y^-) \end{pmatrix}\right\} \\ - \langle\rho(\mathbf{r})\rangle_{\text{Mean}}\frac{2\beta^2 k_L^3\epsilon(\mathbf{r})}{\hbar\epsilon_0 3\pi}\begin{pmatrix} \bar{J}_x(2\tilde{D}_x^+(t')\tilde{D}_x^-(t) + \tilde{D}_y^+(t')\tilde{D}_y^-(t)) \\ \bar{J}_y(\tilde{D}_x^+(t')\tilde{D}_x^-(t) + 2\tilde{D}_y^+(t')\tilde{D}_y^-(t)) \\ \bar{J}_z(\tilde{D}_x^+(t')\tilde{D}_x^-(t) + \tilde{D}_y^+(t')\tilde{D}_y^-(t)) \end{pmatrix}. \quad (7.2.5)$$

These terms may be reduced in a manner similar to equation (7.2.4), where we use the definition of Stokes operators (D.3.1), and the single mode expansion of the field (7.2.2). Detailed calculations

are put in Appendix D.4, the result being:

$$\begin{aligned} & \frac{\langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \beta^2 k_L^3 \epsilon^2(\mathbf{r}) \omega_L}{3\pi} \frac{|u_{\mathbf{k}}(\mathbf{r}, t)|^2}{V} \left\{ \hat{S}_y \begin{pmatrix} \bar{J}_y \\ \bar{J}_x \\ 0 \end{pmatrix} + \hat{S}_z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \\ & - \frac{\langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \beta^2 k_L^3 \epsilon^2(\mathbf{r}) \omega_L}{3\pi} \frac{|u_{\mathbf{k}}(\mathbf{r}, t)|^2}{V} \left\{ \hat{I}_L \begin{pmatrix} \bar{J}_x \\ \bar{J}_y \\ \bar{J}_z \end{pmatrix} + \begin{pmatrix} \bar{J}_x \hat{a}_1^\dagger \hat{a}_1 \\ \bar{J}_y \hat{a}_2^\dagger \hat{a}_2 \\ 0 \end{pmatrix} \right\}. \end{aligned} \quad (7.2.6)$$

We introduced the operator \hat{I}_L . This measures the intensity of the light and is simply give by: $\hat{I}_L = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2$. The terms (7.2.4) and (7.2.6) constitute the first order processes and different types of decay for the atoms. In the next section we will look into the physical reabsorption terms.

7.2.3 Dealing with physical reabsorption terms

Let us first consider the term describing a photon interacting with an atom that previously interacted with a different photon. The term is depicted in Figure 6.6. We consider the term on ordered form, and again use $\tilde{D}_z(\mathbf{r}, t) = 0$. From equation (D.2.1) and the equations (D.2.2), (D.2.3) and (D.2.4), we find the term to reduce to:

$$\begin{aligned} -\langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \frac{4\beta^2}{\hbar^2 \epsilon_0^2} \int_{t_0}^t dt' & \left\{ \left(\begin{array}{c} -\bar{J}_x \left[\tilde{D}_x^+(t') \tilde{D}_x^+(t) \tilde{D}_y^-(t') \tilde{D}_y^-(t) + \tilde{D}_y^+(t) \tilde{D}_y^+(t') \tilde{D}_x^-(t) \tilde{D}_x^-(t') \right] \\ -\bar{J}_y \left[\tilde{D}_x^+(t') \tilde{D}_x^+(t) \tilde{D}_y^-(t') \tilde{D}_y^-(t) + \tilde{D}_y^+(t) \tilde{D}_y^+(t') \tilde{D}_x^-(t) \tilde{D}_x^-(t') \right] \\ 0 \end{array} \right) \right. \\ & \left. + \left(\begin{array}{c} \bar{J}_x \left[\tilde{D}_x^+(t') \tilde{D}_y^+(t) \tilde{D}_y^-(t') \tilde{D}_x^-(t) + \tilde{D}_x^+(t) \tilde{D}_y^+(t') \tilde{D}_y^-(t) \tilde{D}_x^-(t') \right] \\ \bar{J}_y \left[\tilde{D}_x^+(t') \tilde{D}_y^+(t) \tilde{D}_y^-(t') \tilde{D}_x^-(t) + \tilde{D}_x^+(t) \tilde{D}_y^+(t') \tilde{D}_y^-(t) \tilde{D}_x^-(t') \right] \\ 0 \end{array} \right) \right\} \end{aligned} \quad (7.2.7)$$

This equation we may reduce using the expansion (7.2.2). This way we find the following. Detailed calculation are put in Appendix D.5.

$$\frac{\langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \beta^2 \epsilon^2(\mathbf{r}) \omega_L^2}{V} \frac{|u_{\mathbf{k}}(\mathbf{r}, t)|^2}{V} \left[\int_0^t dt' |u_{\mathbf{k}}(\mathbf{r}, t')|^2 \right] \hat{\Gamma}_z^{(2)} \begin{pmatrix} \bar{J}_x \\ \bar{J}_y \\ 0 \end{pmatrix} \quad (7.2.8)$$

We defined the second order Stokes operator $\hat{\Gamma}_z^{(2)}$. This is defined as the following:

$$\hat{\Gamma}_z^{(2)} = \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_2 \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_1 \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1 \quad (7.2.9)$$

As showed in Appendix D.5, the operator $\hat{\Gamma}_z^{(2)}$ can be written in terms of the local Stokes operators, that is $\hat{\Gamma}_z^{(2)} = -\hat{S}_z^2 + \hat{I}_L$.

Before we comment on the different terms let us turn to the remaining type of term, where a photon interacts with an atom, after having interacted with another atom. This term requires a little more calculation. Let us begin with equation (6.4.7). We thus need to know the propagator denoted by \bar{P}^+ . Since this is given in terms of the Green's function \bar{G}^+ in the following way:

$$\bar{P}^+(\mathbf{r}, t | \mathbf{r}', t') = \nabla' \times \nabla' \times \bar{G}^+(\mathbf{r}, t | \mathbf{r}', t'), \quad (7.2.10)$$

and we already established knowledge regarding the Green's function, (6.3.5), finding the propagator is merely a matter of differentiation. The particular differentiation has been put in Appendix D.6. When calculating the propagator we find that this gives rise to a Near-Field and a Far-Field. This is not surprising, since we know from classical electrodynamics that the field originating from a dipole is exactly going to be described as a near- and a far field.[†] Using the far field solution as found in equation (D.6.9), we end up with a term given by the following.

$$\begin{aligned} & \frac{2\beta^2 c^2}{\hbar \epsilon_0} \mathbf{J} \times \left\{ \tilde{\mathbf{D}}^+ \times \left(\bar{P}^- \cdot [\mathbf{J}' \times \tilde{\mathbf{D}}'^-] \right) \right\}_{\text{Far-field}} \\ & = \frac{2c^2 \beta^2}{\hbar \epsilon_0} \int d^3 r' \mathbf{J} \times \left\{ \tilde{\mathbf{D}}_0^+(\mathbf{r}, t) \times \left[4k_L^2 \frac{\mathbf{n} \times [(\langle \rho(\mathbf{r}') \rangle_{\text{Mean}} \mathbf{J}(\mathbf{r}') \times \tilde{\mathbf{D}}_0^-(\mathbf{r}', t') \times \mathbf{n}]}{|\mathbf{r} - \mathbf{r}'|} \right] \right\}, \quad (7.2.11) \end{aligned}$$

were we introduced the unit vector $\mathbf{n} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$. Also we introduced the retarded time t' , this is given as the following: $t' = t - \frac{\sqrt{\epsilon(\mathbf{r})} |\mathbf{r} - \mathbf{r}'|}{c}$. To estimate this term, we need to know something about the scalarfield $\langle \rho(\mathbf{r}) \rangle_{\text{Mean}}$. However this estimation is beyond the scope of this thesis. We note that the one over distance (1/R) dependency is less surprising considering the result found by Michael M. Burns *et al.* (50). In this article they consider the classical interactions between dipoles in an oscillating electric field, and find the interaction energy to have a one over distance dependence.

Likewise we have to consider the Near-field interaction terms generated from equation (6.4.7). Again we refer to Appendix D.6, where the necessary calculations have been put. The result one

[†]See, e.g., (42) page 410 ff.

finds is written below.

$$\begin{aligned}
& - \frac{2\beta^2 c^2 \langle \rho(\mathbf{r}) \rangle_{\text{Mean}}}{\hbar \epsilon_0} \hat{\mathbf{J}} \times \left\{ \left(\bar{\mathbf{P}}^+ \cdot [\hat{\mathbf{J}}' \times \tilde{\mathbf{D}}'^+] \right) \times \tilde{\mathbf{D}}^+ - \tilde{\mathbf{D}}^+ \times \left(\bar{\mathbf{P}}^- \cdot [\hat{\mathbf{J}}' \times \tilde{\mathbf{D}}'^-] \right) \right\}_{\text{Near-field}} \\
& = - \frac{2\beta^2 c^2 \langle \rho(\mathbf{r}) \rangle_{\text{Mean}}}{\hbar \epsilon_0} \hat{\mathbf{J}} \times \left(\int d^3 r' \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \left\{ \left[3(\mathbf{n} \cdot [\langle \rho(\mathbf{r}') \rangle_{\text{Mean}} \bar{\mathbf{J}}(\mathbf{r}') \times \tilde{\mathbf{D}}_0^+(\mathbf{r}', t')]) \mathbf{n} \right. \right. \right. \\
& \quad \left. \left. - [\langle \rho(\mathbf{r}') \rangle_{\text{Mean}} \bar{\mathbf{J}}(\mathbf{r}') \times \tilde{\mathbf{D}}_0^+(\mathbf{r}', t')] \right] e^{ik_L |\mathbf{r} - \mathbf{r}'|} \times \tilde{\mathbf{D}}_0^-(\mathbf{r}, t) \right. \\
& \quad \left. - \tilde{\mathbf{D}}_0^+(\mathbf{r}, t) \times \left[3(\mathbf{n} \cdot [\langle \rho(\mathbf{r}') \rangle_{\text{Mean}} \bar{\mathbf{J}}(\mathbf{r}') \times \tilde{\mathbf{D}}_0^-(\mathbf{r}', t')]) \mathbf{n} \right. \right. \\
& \quad \left. \left. - [\langle \rho(\mathbf{r}') \rangle_{\text{Mean}} \bar{\mathbf{J}}(\mathbf{r}') \times \tilde{\mathbf{D}}_0^-(\mathbf{r}', t')] \right] e^{-ik_L |\mathbf{r} - \mathbf{r}'|} \right\} \\
& \quad + \frac{4ik_L \beta^2 c^2 \langle \rho(\mathbf{r}) \rangle_{\text{Mean}}}{\hbar \epsilon_0} \bar{\mathbf{J}} \times \left(\int d^3 r' \frac{1}{|\mathbf{r} - \mathbf{r}'|^2} \tilde{\mathbf{D}}_0^+(\mathbf{r}, t) \times \left[3(\mathbf{n} \cdot [\langle \rho(\mathbf{r}') \rangle_{\text{Mean}} \bar{\mathbf{J}}(\mathbf{r}') \times \tilde{\mathbf{D}}_0^-(\mathbf{r}', t')]) \mathbf{n} \right. \right. \\
& \quad \left. \left. - [\langle \rho(\mathbf{r}') \rangle_{\text{Mean}} \bar{\mathbf{J}}(\mathbf{r}') \times \tilde{\mathbf{D}}_0^-(\mathbf{r}', t')] \right] e^{-ik_L |\mathbf{r} - \mathbf{r}'|} \right) \tag{7.2.12}
\end{aligned}$$

We suspect these terms to be of small effect, they all comprise the type of integration showed in equation (7.2.13). A simple calculation shows that, when writing the integral in polar coordinates, and integrating out the angular dependence, the integral is seen to vanish.

$$\int d^3 r \frac{1}{|\mathbf{r}|} (3(\mathbf{n} \cdot \mathbf{e}_z) \mathbf{n} - \mathbf{n}) = 0 \tag{7.2.13}$$

Of course our integrations are somewhat more complex, however actually doing the integrations is beyond the scope of this thesis. The terms written in equation (7.2.12) and, in particular, equation (7.2.11), we will refer to as light induced dipole interaction terms. In the next section we sum up the result we have derived, and we discuss the impact of the individual terms entering.

7.3 The final result

In equation (7.3.1), we have written the three dimensional equation of motion for spin at atoms interacting with a single mode field. We have taken the spatial mean over positions of atoms, and divided with $\langle \rho(\mathbf{r}) \rangle_{\text{Mean}}$ on both sides of the equation. The equation is by all means nothing more than a single mode version of equation (6.1.6), taking the spatial mean of the atoms.

$$\frac{d}{dt} \bar{\mathbf{J}}(\mathbf{r}, t) = -\frac{2\beta\omega_L \epsilon(\mathbf{r}) |u_{\mathbf{k}}(\mathbf{r}, t)|^2}{V} \hat{S}_z \begin{pmatrix} \bar{J}_y \\ -\bar{J}_x \\ 0 \end{pmatrix} \quad (7.3.1a)$$

$$+ \frac{\beta^2 k_L^3 \epsilon^2(\mathbf{r}) \omega_L |u_{\mathbf{k}}(\mathbf{r}, t)|^2}{3\pi V} \left\{ \hat{S}_y \begin{pmatrix} \bar{J}_y \\ \bar{J}_x \\ 0 \end{pmatrix} + \hat{S}_z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad (7.3.1b)$$

$$- \frac{\beta^2 k_L^3 \epsilon^2(\mathbf{r}) \omega_L |u_{\mathbf{k}}(\mathbf{r}, t)|^2}{3\pi V} \left\{ \hat{I}_L \begin{pmatrix} \bar{J}_x \\ \bar{J}_y \\ \bar{J}_z \end{pmatrix} + \begin{pmatrix} \bar{J}_x \hat{a}_1^\dagger \hat{a}_1 \\ \bar{J}_y \hat{a}_2^\dagger \hat{a}_2 \\ 0 \end{pmatrix} \right\} \quad (7.3.1c)$$

$$+ \frac{\beta^2 \epsilon^2(\mathbf{r}) \omega_L^2 |u_{\mathbf{k}}(\mathbf{r}, t)|^2}{V} \left[\int_0^t dt' |u_{\mathbf{k}}(\mathbf{r}, t')|^2 \right] \hat{\Gamma}_z^{(2)} \begin{pmatrix} \bar{J}_x \\ \bar{J}_y \\ 0 \end{pmatrix} \quad (7.3.1d)$$

+ Light induced dipole interaction terms.

Let us consider the first term (7.3.1a). This term is best described as a rotation of the spin. The operator \hat{S}_z when expanded in terms of a rotation basis, exactly describes the intensity of an incoming light beam of either σ_+ or σ_- polarization. Assuming the spin of the atom to be oriented along the x - or the y -axis, the angular momentum of the light will then cause the spin of the atom to rotate. (29; 30; 32) The second term (7.3.1b) is comprised of two parts. The first part includes the Stokes operator \hat{S}_y this operator can be expanded in a basis measuring either photons linearly polarized along an axis of $+45^\circ$ or -45° ,

$$\hat{S}_y = \frac{1}{2} (\hat{a}_{+45^\circ}^\dagger \hat{a}_{+45^\circ} - \hat{a}_{-45^\circ}^\dagger \hat{a}_{-45^\circ}). \quad (7.3.2)$$

In this respect we find that if the spin was originally oriented along the x -axis it will rotate toward the y -axis. The second part of (7.3.1b), concerns the z -component of the spin operator. This term describes how a scattering process may increase or decrease the spin along the quantization axis. Consider the Stokes operator \hat{S}_z expanded in rotating basis:

$$\hat{S}_z = \frac{1}{2} (\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-). \quad (7.3.3)$$

In Figure 7.2 is showed scattering process, where the incoming field is σ_+ polarized. In this case an excited atom can either decay to the $m = +\frac{1}{2}$ state, as showed in the figure, or to the

$m = -\frac{1}{2}$ state, in which case nothing happens. On average this will increase the spin. We note that in the case where all atoms in the gas are prepared in the state $m = +\frac{1}{2}$, there will be no interactions between σ_+ polarized light and the atoms, a phenomenon known as optically induced transparency. (51; 52; 53) For σ_- polarized light, the opposite process happens, and we experience a decrease of spin along the z -axis. The term (7.3.1c) describes a general decay of the spin. We

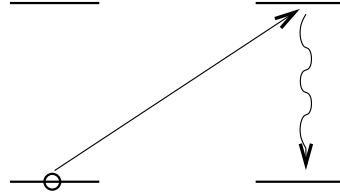


Figure 7.2: An atom absorbs a σ_+ polarized photon, this process increases the atomic spin

notice that if we send in linearly polarized light along the x -axis, we will find the spin in the x -axis to decay at twice the rate of both the y component of the spin and the z component of the spin. The last term we comment on is (7.3.1d). This term describes, as previously discussed, the process of two consecutive light interactions with the atom. Naively guessed one could imagine that the interaction described in term (7.3.1a) should just be applied twice. We could write the interaction as:

$$\gamma \hat{S}_z \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \bar{J}_x \\ \bar{J}_y \end{bmatrix}. \quad (7.3.4)$$

We consider only the spin components involved, hence the \bar{J}_x and \bar{J}_y . Since the anti-symmetric matrix, when squared, equals minus the identity, we would, even on that account, expect a term similar to (7.3.1d). This equation ends the calculations in this thesis. In the following chapter we conclude on the work done in the thesis, as well as the work to be done in the future. [‡]

[‡]To the keen reader: Have no worries, the story will continue

Part IV

Conclusion

Conclusion

We took upon us the task of deriving a three-dimensional theory describing light-atom interactions. Different groups have each given their point of view on the matter (see, e.g., (34; 41; 25)). A real satisfactory theory is yet to be seen as noted by H.J. Müller *et al.*: “This work ... *may serve as a starting point for a more elaborate quantum theoretical treatment*”(34). This thesis was motivated by a desire, as well as a need, to derive a more elaborate quantum theory for light-atom interactions.

The theory derived in this master thesis is firmly based on mathematical reasoning. Taught by experience* we started of at an almost field theoretical footing. Our line of approach was firstly to make sure that classical mechanics was derivable, i.e., macroscopic Maxwell equations. Regarding the quantization process there followed, we kept in mind effects that could be considered classical of nature. It is never a trivial task to quantize a theory, and one should ask oneself what to understand by a quantized theory. We wanted to clarify both the canonical quantization process, defining commutationrelations on the fields, but also the microscopic quantization process, defining creation and annihilation operators. In this respect we derived in Chapter 4 a quantum theory where the range of usability reached from classical effects such as diffraction, to highly nontrivial quantum mechanical effects such as spontaneous emission of single photons.

In Chapter 5 we derived a set of equations of motion for our system, and we showed how to separate the diffracting part of the equations from the microscopic part. When changing to the slowly oscillating picture we met another challenge. We abandoned time symmetries in the

*See discussion section 2.5.

system evolution, in other words we changed from a wave equation to a diffusion equation. On this account we had to derive methods for solving the set of three dimensional diffusion equations. Thus mathematically tested we reached in Chapter 6 the core of quantum mechanics. Having established methods of dealing with the randomness of position of atoms, we payed particular interest on the time-evolution of the atoms. In these calculations we found the effect of nontrivial commutationrelations both regarding the light field, but also the spin of the atoms. We found decay processes that only quantum mechanics could discover. Finally in Chapter 7 we cast our equation regarding the spin of the atoms in a single mode form this in preparation for experimental applications.

The theory so far developed, has given us a powerful tool for considering light interacting with ensembles of atoms. In the introduction we described some of the unsolved puzzles regarding ensembles. Though we have not solved these problems, we have developed quantitative expressions concerning, e.g., light induced dipole interactions. We found in section 2.4 page 22 that to lowest order, two ensembles could be entangled, to a degree depending, among other things, on the number of atoms in the ensemble. One could imagine that by introducing more atoms to the ensemble, the effect of light induced dipole interactions would increase. The ultimate question is thus, will we reach a point where the gain by adding more atoms is balanced by the loss due to light induced dipole interactions and decay?

Apart from being the first real three-dimensional theory of light interacting with an atomic ensemble this thesis also sets the stage for future work like, e.g., the estimation of induced-dipole interactions, high-order equations for the Stokes operator, etc. Similarly, it is an open question how to give a detailed three-dimensional mapping protocol between quantum information stored in light and atomic ensembles.

The usability of ensembles as quantum computers is intimately connected with our ability to transfer quantum information from light to ensembles and vice versa. The advent of quantum computing will therefore boost the research on light-matter interactions and we expect the topics discussed in this thesis to receive a constantly-increasing attention. Put otherwise, *novel, powerful and complete as it may seem ... quantum computing ... is feasible only to the extend on which we may gain access to quantum-like systems.*

Commutationrelations for Π^+ , Π^-

We know the equal-time commutator for the full $\hat{\Pi}$ - and \hat{A} -field.

$$[\hat{A}_m(\mathbf{r}, t); \hat{\Pi}_m(\mathbf{r}', t)] = i\hbar\delta(\mathbf{r} - \mathbf{r}')$$

We are now interested in finding commutationrelations for \hat{A} and the positive as well as the negative part of the $\hat{\Pi}$ -field. We have:

$$\begin{aligned} i\hbar\delta^3(\mathbf{r} - \mathbf{r}') &= [\hat{A}_m(\mathbf{r}, t); \hat{\Pi}_m(\mathbf{r}', t)] = [\hat{A}_m(\mathbf{r}, t); \hat{\Pi}_m^+(\mathbf{r}', t) + \hat{\Pi}_m^-(\mathbf{r}', t)] \\ &= [\hat{A}_m(\mathbf{r}, t); \hat{\Pi}_m^+(\mathbf{r}', t)] + [\hat{A}_m(\mathbf{r}, t); \hat{\Pi}_m^-(\mathbf{r}', t)] \\ &\equiv \hat{f}(\mathbf{r}, \mathbf{r}') + \hat{g}(\mathbf{r}, \mathbf{r}') \end{aligned} \tag{A.1.1}$$

We further more see that:

$$\begin{aligned} \hat{f}^\dagger(\mathbf{r}, \mathbf{r}') &= [\hat{A}_m(\mathbf{r}, t); \hat{\Pi}_m^+(\mathbf{r}', t)]^\dagger = (\hat{A}_m(\mathbf{r}, t)\hat{\Pi}_m^+(\mathbf{r}', t) - \hat{\Pi}_m^+(\mathbf{r}', t)\hat{A}_m(\mathbf{r}, t))^\dagger \\ &= \hat{\Pi}_m^-(\mathbf{r}', t)\hat{A}_m(\mathbf{r}, t) - \hat{A}_m(\mathbf{r}, t)\hat{\Pi}_m^-(\mathbf{r}', t) = -[\hat{A}_m(\mathbf{r}, t); \hat{\Pi}_m^-(\mathbf{r}', t)] \\ &= -\hat{g}(\mathbf{r}, \mathbf{r}') \end{aligned} \tag{A.1.2}$$

Let $\hat{H}(\mathbf{r})$ be a hermitian operator, we then know that:

$$\int d^3r \hat{H}(\mathbf{r}) (\hat{f}(\mathbf{r}, \mathbf{r}') + \hat{g}(\mathbf{r}, \mathbf{r}')) = \int d^3r \hat{H}(\mathbf{r}) i\hbar\delta^3(\mathbf{r} - \mathbf{r}') = i\hbar\hat{H}(\mathbf{r}') \tag{A.1.3}$$

We then see that $\forall \mathbf{r}'$ we must have according to equation A.1.3:

$$\int d^3 r \hat{H}(\mathbf{r}) \hat{f}(\mathbf{r}, \mathbf{r}') = \hat{\alpha}(\mathbf{r}') \quad (\text{A.1.4})$$

$$\int d^3 r \hat{H}(\mathbf{r}) \hat{g}(\mathbf{r}, \mathbf{r}') = i\hbar \hat{H}(\mathbf{r}') - \hat{\alpha}(\mathbf{r}') \quad (\text{A.1.5})$$

Which again means:

$$\begin{aligned} \hat{\alpha}(\mathbf{r}')^\dagger &= \left(\int d^3 r \hat{H}(\mathbf{r}) \hat{f}(\mathbf{r}, \mathbf{r}') \right)^\dagger = - \left(\int d^3 r \hat{H}(\mathbf{r}) \hat{g}(\mathbf{r}, \mathbf{r}') \right) \\ &= \hat{\alpha}(\mathbf{r}') - i\hbar \hat{H}(\mathbf{r}') \end{aligned} \quad (\text{A.1.6})$$

We therefore conclude that:

$$\text{Im}(\hat{\alpha}(\mathbf{r}')) = \frac{\hbar}{2} \hat{H}(\mathbf{r}') \quad (\text{A.1.7})$$

We can therefore allow for:

$$\hat{f}(\mathbf{r}, \mathbf{r}') = \hat{k}(\mathbf{r}, \mathbf{r}') + i\frac{\hbar}{2} \delta^3(\mathbf{r} - \mathbf{r}') \quad (\text{A.1.8})$$

$$\hat{g}(\mathbf{r}, \mathbf{r}') = -\hat{k}(\mathbf{r}, \mathbf{r}') + i\frac{\hbar}{2} \delta^3(\mathbf{r} - \mathbf{r}') \quad (\text{A.1.9})$$

Where $\hat{k}^\dagger(\mathbf{r}, \mathbf{r}') = \hat{k}(\mathbf{r}, \mathbf{r}')$. Turning back to our definition of \hat{f} we see:

$$\begin{aligned} \hat{f}(\mathbf{r}, \mathbf{r}') &= [\hat{A}_m(\mathbf{r}, t); \hat{\Pi}_m^+(\mathbf{r}', t)] = \hat{k}(\mathbf{r}, \mathbf{r}') + i\frac{\hbar}{2} \delta^3(\mathbf{r} - \mathbf{r}') \\ &= i [\hat{A}_m(\mathbf{r}, t); O[\hat{\Pi}(\mathbf{r}', t)]_w] + \frac{1}{2} [\hat{A}_m(\mathbf{r}, t); \hat{\Pi}_m(\mathbf{r}', t)] \end{aligned} \quad (\text{A.1.10})$$

Where $O[\hat{\Pi}(\mathbf{r}', t)]$ is some functional of $\hat{\Pi}$. We see that one can make a transformation on $\hat{\Pi}^+$ and $\hat{\Pi}^-$ without changing the physically relevant field $\hat{\Pi}$:

$$\hat{\Pi}^+(\mathbf{r}, t) \rightarrow \hat{\Pi}^+'(\mathbf{r}, t) = \hat{\Pi}^+(\mathbf{r}, t) + iO[\hat{\Pi}(\mathbf{r}', t)] \quad (\text{A.1.11})$$

$$\hat{\Pi}^-(\mathbf{r}, t) \rightarrow \hat{\Pi}^-('(\mathbf{r}, t) = \hat{\Pi}^-(\mathbf{r}, t) - iO[\hat{\Pi}(\mathbf{r}', t)] \quad (\text{A.1.12})$$

From this symmetry we see that it is always possible to have $\hat{\Pi}^+$ and $\hat{\Pi}^-$ such that $\hat{k}(\mathbf{r}, \mathbf{r}') = 0$. We then arrive at the simple commutationrelations for $\hat{\Pi}^+$ and $\hat{\Pi}^-$:

$$[\hat{A}_m(\mathbf{r}, t); \hat{\Pi}_n^+(\mathbf{r}', t)] = [\hat{A}_m(\mathbf{r}, t); \hat{\Pi}_n^-(\mathbf{r}', t)] = i\hbar \frac{1}{2} \delta^3(\mathbf{r} - \mathbf{r}') \delta_{nm} \quad (\text{A.1.13})$$

Detailed Calculations, Part I

B.1.1 Defining rotating basis

The basis vectors in the rotating basis are

$$\mathbf{e}_+ = \frac{\mathbf{e}_x + i\mathbf{e}_y}{\sqrt{2}}, \quad \mathbf{e}_- = \frac{\mathbf{e}_x - i\mathbf{e}_y}{\sqrt{2}}, \quad \mathbf{e}_0 = \mathbf{e}_z, \quad (\text{B.1.1})$$

$\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ being the usual Euclidean basis vectors. Scalar products and cross products between the basisvectors is shown to be

$$\begin{aligned} \mathbf{e}_+ \cdot \mathbf{e}_- &= 1 & \mathbf{e}_- \cdot \mathbf{e}_+ &= 1 & \mathbf{e}_0 \cdot \mathbf{e}_0 &= 1 \\ \mathbf{e}_+ \times \mathbf{e}_- &= -i\mathbf{e}_0 & \mathbf{e}_- \times \mathbf{e}_+ &= i\mathbf{e}_0 & \mathbf{e}_+ \times \mathbf{e}_0 &= i\mathbf{e}_+ \\ \mathbf{e}_0 \times \mathbf{e}_+ &= -i\mathbf{e}_- & \mathbf{e}_- \times \mathbf{e}_0 &= -i\mathbf{e}_- & \mathbf{e}_0 \times \mathbf{e}_- &= i\mathbf{e}_- \end{aligned} \quad (\text{B.1.2})$$

B.2 Regarding the Effective Hamiltonian

B.2.1 Dipole operator

We wrote the dipole operator as:

$$\mathbf{d} = V_{-1}^{(1)} \mathbf{e}_+ - V_{+1}^{(1)} \mathbf{e}_- + V_0^{(1)} \mathbf{e}_0 \quad (\text{B.2.1})$$

Thus the Matrix elements we are interested in calculating are:

$$\langle n|V_q^{(1)}|m\rangle \quad \text{for } q \in \{-1, 0, +1\}, n, m \in \{1, 2, 3, 4\} \quad (\text{B.2.2})$$

The Wigner-Eckart theorem states the following* *The matrix elements of tensor operators with respect to angular momentum eigenstates satisfy*

$$\langle \alpha', j' m' | T_q^{(k)} | \alpha, j m \rangle = \langle j k; m q | j k; j' m' \rangle \frac{\langle \alpha' j' || T^{(k)} || \alpha j \rangle}{\sqrt{2j+1}} \quad (\text{B.2.3})$$

where the double-bar matrix element is independent of m and m' , and q . Using this theorem we may calculate the matrix element $\langle 1|V_{-1}^1|3\rangle$.

$$\begin{aligned} \langle 1|V_{-1}^1|3\rangle &= \langle g, \frac{1}{2}, -\frac{1}{2} | V_{-1}^1 | e, \frac{1}{2}, \frac{1}{2} \rangle \\ &= \langle \frac{1}{2} 1; \frac{1}{2} - 1 | \frac{1}{2} 1; \frac{1}{2} - \frac{1}{2} \rangle \frac{\langle g \frac{1}{2} || V^1 || e \frac{1}{2} \rangle}{\sqrt{2\frac{1}{2} + 1}} \\ &= \sqrt{\frac{2}{3}} \frac{\langle g \frac{1}{2} || V^1 || e \frac{1}{2} \rangle}{\sqrt{2\frac{1}{2} + 1}}. \end{aligned}$$

Setting $\langle g \frac{1}{2} || V^1 || e \frac{1}{2} \rangle = \sqrt{3}gd$, we find,

$$=gd \quad (\text{B.2.4})$$

Similarly one may calculate the remaining matrix elements and find the following:

$$\begin{aligned} \langle 1|\hat{V}_{-1}^1|3\rangle &= gd, & \langle 3|\hat{V}_{+1}^1|1\rangle &= -gd, \\ \langle 4|\hat{V}_{-1}^1|2\rangle &= gd, & \langle 2|\hat{V}_{+1}^1|4\rangle &= -gd, \\ \langle 1|\hat{V}_0^1|4\rangle &= -\frac{1}{\sqrt{2}}gd, & \langle 4|\hat{V}_0^1|1\rangle &= -\frac{1}{\sqrt{2}}gd, \\ \langle 3|\hat{V}_0^1|2\rangle &= \frac{1}{\sqrt{2}}gd, & \langle 2|\hat{V}_0^1|3\rangle &= \frac{1}{\sqrt{2}}gd, \end{aligned} \quad (\text{B.2.5})$$

*See the book (35) for excellent derivation

B.2.2 Remarks on the electric field

Writing the electric field in the Euclidean basis, and then changing to the rotating basis, we find:

$$\begin{aligned}
\hat{\mathbf{E}}^+(\mathbf{r}, t) &= \hat{E}_x^+ \mathbf{e}_x + \hat{E}_y^+ \mathbf{e}_y + \hat{E}_z^+ \mathbf{e}_z \\
&= \left(\frac{\hat{E}_x^+ - i\hat{E}_y^+}{\sqrt{2}} \right) \left(\frac{\mathbf{e}_x + i\mathbf{e}_y}{\sqrt{2}} \right) + \left(\frac{\hat{E}_x^+ + i\hat{E}_y^+}{\sqrt{2}} \right) \left(\frac{\mathbf{e}_x - i\mathbf{e}_y}{\sqrt{2}} \right) + \hat{E}_z^+ \mathbf{e}_z \\
&= \hat{E}_-^+ \mathbf{e}_+ + \hat{E}_+^+ \mathbf{e}_- + \hat{E}_0^+ \mathbf{e}_0,
\end{aligned} \tag{B.2.6}$$

where we defined

$$\hat{E}_-^+ = \frac{\hat{E}_x^+ - i\hat{E}_y^+}{\sqrt{2}}, \quad \hat{E}_+^+ = \frac{\hat{E}_x^+ + i\hat{E}_y^+}{\sqrt{2}}, \quad \hat{E}_0^+ = \hat{E}_z^+. \tag{B.2.7}$$

To shorten the notation we did not write explicitly the space and time dependence on the coefficients this is implied. Similarly we write the negatively oscillating part of the field in the rotation basis

$$\hat{\mathbf{E}}^-(\mathbf{r}, t) = \hat{E}_-^- \mathbf{e}_+ + \hat{E}_+^- \mathbf{e}_- + \hat{E}_0^- \mathbf{e}_0 \tag{B.2.8}$$

where we likewise defined

$$\hat{E}_-^- = \frac{\hat{E}_x^- - i\hat{E}_y^-}{\sqrt{2}}, \quad \hat{E}_+^- = \frac{\hat{E}_x^- + i\hat{E}_y^-}{\sqrt{2}}, \quad \hat{E}_0^- = \hat{E}_z^-. \tag{B.2.9}$$

B.3 Adiabatic elimination

Below is written the expressions for the atomic coherence operators, after adiabatic calculations.

$$\begin{aligned}
\hat{\sigma}_{24} &= \frac{gd}{\hbar\Delta} (\hat{E}_+^- \hat{\sigma}_{22}^j - \hat{E}_0^- \hat{J}_+) & \hat{\sigma}_{13} &= \frac{gd}{\hbar\Delta} (\hat{E}_-^- \hat{\sigma}_{11}^j + \hat{E}_0^- \hat{J}_-) \\
\hat{\sigma}_{42} &= \frac{gd}{\hbar\Delta} (\hat{E}_+^+ \hat{\sigma}_{22}^j - \hat{E}_0^+ \hat{J}_-) & \hat{\sigma}_{31} &= \frac{gd}{\hbar\Delta} (\hat{E}_+^+ \hat{\sigma}_{11}^j + \hat{E}_0^+ \hat{J}_+) \\
\hat{\sigma}_{23} &= \frac{gd}{\hbar\Delta} \left(\frac{1}{\sqrt{2}} \hat{E}_0^- \hat{\sigma}_{22}^j + \sqrt{2} \hat{E}_-^- \hat{J}_+ \right) & \hat{\sigma}_{14} &= -\frac{gd}{\hbar\Delta} \left(\frac{1}{\sqrt{2}} \hat{E}_0^- \hat{\sigma}_{11}^j - \sqrt{2} \hat{E}_+^- \hat{J}_- \right) \\
\hat{\sigma}_{32} &= \frac{gd}{\hbar\Delta} \left(\frac{1}{\sqrt{2}} \hat{E}_0^+ \hat{\sigma}_{22}^j + \sqrt{2} \hat{E}_+^+ \hat{J}_- \right) & \hat{\sigma}_{41} &= -\frac{gd}{\hbar\Delta} \left(\frac{1}{\sqrt{2}} \hat{E}_0^+ \hat{\sigma}_{11}^j - \sqrt{2} \hat{E}_-^+ \hat{J}_+ \right).
\end{aligned} \tag{B.3.1}$$

B.3.1 Remarks on the reduction of the Hamiltonian, equation (2.2.24)

Below follows the reduction of the effective Hamiltonian given in equation (2.2.23) to the final effective Hamiltonian given in equation (2.2.24)

$$\begin{aligned}
\hat{\mathcal{H}}_{\text{Eff}} &= - \sum_j^{\text{Atoms}} \frac{g^2 d^2}{\hbar \Delta} \left\{ \hat{E}_-^+ (\hat{E}_+^- \hat{\sigma}_{22}^j - \hat{E}_0^- \hat{J}_+^j) + \hat{E}_-^- (\hat{E}_+^+ \hat{\sigma}_{11}^j + \hat{E}_0^+ \hat{J}_+^j) + \hat{E}_+^+ (\hat{E}_-^- \hat{\sigma}_{11}^j + \hat{E}_0^- \hat{J}_-^j) \right. \\
&\quad + \hat{E}_+^- (\hat{E}_-^+ \hat{\sigma}_{22}^j - \hat{E}_0^+ \hat{J}_-^j) + \frac{1}{\sqrt{2}} \left[\hat{E}_0^+ \left(\frac{1}{\sqrt{2}} \hat{E}_-^- \hat{\sigma}_{22}^j + \sqrt{2} \hat{E}_-^- \hat{J}_+^j \right) + \hat{E}_0^- \left(\frac{1}{\sqrt{2}} \hat{E}_+^+ \hat{\sigma}_{22}^j + \sqrt{2} \hat{E}_+^+ \hat{J}_-^j \right) \right. \\
&\quad \left. \left. - \hat{E}_0^+ \left(\frac{1}{\sqrt{2}} \hat{E}_-^- \hat{\sigma}_{11}^j - \sqrt{2} \hat{E}_+^+ \hat{J}_-^j \right) - \hat{E}_0^- \left(\frac{1}{\sqrt{2}} \hat{E}_+^+ \hat{\sigma}_{11}^j - \sqrt{2} \hat{E}_-^- \hat{J}_+^j \right) \right] \right\} \\
&= - \sum_j^{\text{Atoms}} \frac{g^2 d^2}{\hbar \Delta} \left\{ \hat{E}_-^+ \sigma_{22}^j \hat{E}_+^- - \hat{E}_-^- \hat{J}_+^j \hat{E}_0^- + \hat{E}_+^+ \sigma_{11}^j \hat{E}_-^- + \hat{E}_0^+ \hat{J}_+^j \hat{E}_-^- + \hat{E}_+^+ \sigma_{11}^j \hat{E}_-^- + \hat{E}_+^- \hat{J}_-^j \hat{E}_0^- \right. \\
&\quad + \hat{E}_+^- \sigma_{22}^j \hat{E}_+^- - \hat{E}_0^+ \hat{J}_-^j \hat{E}_+^- + \frac{1}{2} \hat{E}_0^+ \sigma_{22}^j \hat{E}_0^- + \hat{E}_0^+ \hat{J}_+^j \hat{E}_-^- + \frac{1}{2} \hat{E}_0^+ \sigma_{22}^j \hat{E}_0^- + \hat{E}_+^- \hat{J}_-^j \hat{E}_0^- \\
&\quad \left. + \frac{1}{2} \hat{E}_0^+ \sigma_{11}^j \hat{E}_0^- - \hat{E}_0^+ \hat{J}_-^j \hat{E}_+^- + \frac{1}{2} \hat{E}_0^+ \sigma_{11}^j \hat{E}_0^- - \hat{E}_-^- \hat{J}_+^j \hat{E}_0^- \right\} \\
&= - \sum_j^{\text{Atoms}} \frac{g^2 d^2}{\hbar \Delta} \left\{ 2 \hat{E}_-^+ \sigma_{22}^j \hat{E}_+^- + 2 \hat{E}_+^+ \sigma_{11}^j \hat{E}_-^- + \hat{E}_0^+ \sigma_{22}^j \hat{E}_0^- + \hat{E}_0^+ \sigma_{11}^j \hat{E}_0^- - 2 \hat{E}_+^- \hat{J}_+^j \hat{E}_0^- \right. \\
&\quad \left. + 2 \hat{E}_0^+ \hat{J}_+^j \hat{E}_-^- + 2 \hat{E}_+^- \hat{J}_-^j \hat{E}_0^- - 2 \hat{E}_+^- \hat{J}_-^j \hat{E}_+^- \right\} \\
&= - \sum_j^{\text{Atoms}} \frac{g^2 d^2}{\hbar \Delta} \left\{ \hat{E}_-^+ (\sigma_{22}^j + \sigma_{11}^j) \hat{E}_+^- + \hat{E}_-^- (\sigma_{22}^j - \sigma_{11}^j) \hat{E}_+^- + \hat{E}_+^+ (\sigma_{22}^j + \sigma_{11}^j) \hat{E}_-^- - \hat{E}_+^- (\sigma_{22}^j - \sigma_{11}^j) \hat{E}_-^- \right. \\
&\quad \left. + \hat{E}_0^+ (\sigma_{22}^j + \sigma_{11}^j) \hat{E}_0^- - 2 \hat{E}_+^- \hat{J}_+^j \hat{E}_0^- + 2 \hat{E}_0^+ \hat{J}_+^j \hat{E}_-^- + 2 \hat{E}_+^- \hat{J}_-^j \hat{E}_0^- - 2 \hat{E}_0^+ \hat{J}_-^j \hat{E}_+^- \right\} \\
&= - \sum_j^{\text{Atoms}} \frac{2g^2 d^2}{\hbar \Delta} \left\{ \hat{\mathbf{E}}^+ \left(\frac{1}{2} \bar{\mathbf{I}}^j \right) \hat{\mathbf{E}}^- + \hat{E}_-^+ \hat{J}_0^j \hat{E}_+^- - \hat{E}_+^+ \hat{J}_0^j \hat{E}_-^- \right. \\
&\quad \left. - \hat{E}_+^- \hat{J}_+^j \hat{E}_0^- + \hat{E}_0^+ \hat{J}_+^j \hat{E}_-^- + \hat{E}_+^- \hat{J}_-^j \hat{E}_0^- - \hat{E}_0^+ \hat{J}_-^j \hat{E}_+^- \right\} \\
&= - \sum_j^{\text{Atoms}} \frac{2g^2 d^2}{\hbar \Delta} \hat{\mathbf{E}}^+(\mathbf{r}_j, t) \left\{ \frac{1}{2} \bar{\mathbf{I}}^j - i \mathbf{J}^j \times \right\} \hat{\mathbf{E}}^-(\mathbf{r}_j, t) \tag{B.3.2}
\end{aligned}$$

Where we in the last step used equation (B.1.2).

B.4 On the 1-dimensional Theory

B.4.1 Continuous Operator

Below follows a more detailed calculation of the continuous field operator in one dimension, used in equation (2.3.14)

$$\mathbf{E}(\mathbf{r}, t) = \sum_s \sum_k i \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} (\mathbf{e}_s \hat{a}_{ks}(t) e^{ikz} - \mathbf{e}_s^* \hat{a}_{ks}^\dagger(t) e^{-ikz})$$

This we want to change to a continuous operator.

For a plane wave propagating along the z -axis in an empty cavity of length L , parallel to the z -axis, the eigenmodes are separated by $\delta k = \frac{2\pi}{L}$. The continuation of modes we define as:

$$\hat{a}_{ks}(t) \rightarrow \sqrt{\delta k} \hat{a}_s(k, t), \quad (\text{B.4.1})$$

and similarly for $\hat{a}_{sk}^\dagger(t)$. Similarly we have relation

$$\delta_{kk'} = \delta k \delta(k - k') \quad (\text{B.4.2})$$

thus converting the usual commutation relation to

$$[\hat{a}_s(k, t); \hat{a}_{s'}^\dagger(k', t)] = \delta_{ss'} \delta(k - k'). \quad (\text{B.4.3})$$

The sum over discrete quantities becomes an integral,

$$\sum_k \rightarrow \frac{1}{\delta k} \int_0^\infty dk \quad (\text{B.4.4})$$

Using these substitutions we find for the electric field,

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= \sum_s \sum_k i \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} (\mathbf{e}_s \hat{a}_{ks}(t) e^{ikz} - \mathbf{e}_s^* \hat{a}_{ks}^\dagger(t) e^{-ikz}) \\ &= \sum_s \sum_k \delta k i \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 A}} (\mathbf{e}_s \hat{a}_s(k, t) e^{ikz} - \mathbf{e}_s^* \hat{a}_s^\dagger(k, t) e^{-ikz}) \\ &= \sum_s \int_0^\infty dki \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 A}} (\mathbf{e}_s \hat{a}_s(k, t) e^{ikz} - \mathbf{e}_s^* \hat{a}_s^\dagger(k, t) e^{-ikz}) \end{aligned} \quad (\text{B.4.5})$$

Considering a real laser field with a carrier frequency ω_L , allows us to expand the integration to go from $-\infty$ to ∞ If we define the following Fourier transform,

$$\hat{a}_s(z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \hat{a}_s(k, t) e^{ikz} \quad \text{and} \quad \hat{a}_s^\dagger(z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \hat{a}_s^\dagger(k, t) e^{-ikz}, \quad (\text{B.4.6})$$

we finally get:

$$\mathbf{E}(\mathbf{r}, t) = \sum_s i \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 A}} (\mathbf{e}_s \hat{a}_s(z, t) - \mathbf{e}_s^* \hat{a}_s^\dagger(z, t)) \quad (\text{B.4.7})$$

The commutation relations applying to these Fourier transformed creation and annihilation operators are calculated below.

$$\begin{aligned} [\hat{a}_s(z, t); \hat{a}_{s'}^\dagger(z', t)] &= \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \hat{a}_s(k, t) e^{ikz}; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk' \hat{a}_{s'}^\dagger(k', t) e^{-ik'z'} \right] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' [\hat{a}_s(k, t); \hat{a}_{s'}^\dagger(k', t)] e^{ikz} e^{-ik'z'} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \delta_{ss'} \delta(k - k') e^{ikz} e^{-ik'z'} \\ &= \delta_{ss'} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ik(z-z')} \\ &= \delta_{ss'} \delta(z - z') \end{aligned} \quad (\text{B.4.8})$$

B.4.2 Commutation relation on the Atomic spin Operator

The atomic spin operator is defined in equation (2.2.19). In the following we show the commutation relations to which it applies.

$$\hat{J}_x^j = \frac{\hat{J}_+^j + \hat{J}_-^j}{\sqrt{2}}, \quad \hat{J}_y^j = \frac{\hat{J}_+^j - \hat{J}_-^j}{i\sqrt{2}}, \quad \hat{J}_z^j = \hat{J}_0^j \quad (\text{B.4.9})$$

And by simple calculation we see,

$$\begin{aligned} [\hat{J}_x^j; \hat{J}_y^j] &= \frac{1}{4i} [\sigma_{21}^j + \sigma_{12}^j; \sigma_{21}^j - \sigma_{12}^j] \\ &= \frac{1}{4i} \{ [\sigma_{21}^j; \sigma_{21}^j] - [\sigma_{21}^j; \sigma_{12}^j] + [\sigma_{12}^j; \sigma_{21}^j] - [\sigma_{12}^j; \sigma_{12}^j] \} \\ &= \frac{1}{4i} \{ -\sigma_{22}^j + \sigma_{11}^j + \sigma_{11}^j - \sigma_{22}^j \} \\ &= i\hat{J}_z^j \end{aligned} \quad (\text{B.4.10})$$

Similar calculation shows the remaining commutation relations, and we conclude:

$$[\hat{J}_n^j, \hat{J}_m^{j'}] = i\epsilon_{nml}\hat{J}_l^j\delta_{jj'} \quad (\text{B.4.11})$$

We assume different atoms to be uncorrelated, and obviously commuting.

Commutation relations regarding equation (2.3.7)

Commutation relations regarding the field formulation of the atomic spin, can then be found to give the following.

$$\begin{aligned} [\hat{J}_m(\mathbf{r}); \hat{J}_n(\mathbf{r}')] &= \left[\sum_j \hat{J}_n^j \delta(\mathbf{r} - \mathbf{r}_j); \sum_{j'} \hat{J}_m^{j'} \delta(\mathbf{r}' - \mathbf{r}_{j'}) \right] \\ &= \sum_{jj'} \delta(\mathbf{r} - \mathbf{r}_j) \delta(\mathbf{r}' - \mathbf{r}_{j'}) [\hat{J}_m^j; \hat{J}_n^{j'}] \\ &= \sum_{jj'} \delta(\mathbf{r} - \mathbf{r}_j) \delta(\mathbf{r}' - \mathbf{r}_{j'}) i\epsilon_{nml} \hat{J}_l^j \delta_{jj'} \\ &= \sum_j \delta(\mathbf{r} - \mathbf{r}_j) \delta(\mathbf{r}' - \mathbf{r}_j) i\epsilon_{nml} \hat{J}_l^j \\ &= i\delta(\mathbf{r} - \mathbf{r}') \epsilon_{nml} \sum_j \delta(\mathbf{r} - \mathbf{r}_j) \hat{J}_l^j = i\delta(\mathbf{r} - \mathbf{r}') \epsilon_{nml} \hat{J}_l(\mathbf{r}) \end{aligned} \quad (\text{B.4.12})$$

We used the substitution $\delta(\mathbf{r} - \mathbf{r}_j) \delta(\mathbf{r}' - \mathbf{r}_j) = \delta(\mathbf{r} - \mathbf{r}') \delta(\mathbf{r} - \mathbf{r}_j)$. This can be seen by simple testing with arbitrary test function, and then integration.

Commutation relation regarding equation (2.3.9)

The spin operator is defined as below.

$$\hat{J}_n(z, t) \equiv \lim_{\delta z \rightarrow 0} \int_z^{z+\delta z} dz' \iint dx' dy' \frac{1}{\rho A \delta z} \hat{J}_n(\mathbf{r}', t) \quad (\text{B.4.13})$$

We then calculate the commutation relations to the following.

$$\begin{aligned}
[\hat{J}_n(z, t); \hat{J}_m(z', t)] &= \left[\lim_{\delta z \rightarrow 0} \int_z^{z+\delta z} dz'' \iint dx'' dy'' \frac{1}{\rho A \delta z} \hat{J}_n(\mathbf{r}'', t); \right. \\
&\quad \left. \lim_{\delta z' \rightarrow 0} \int_{z'}^{z'+\delta z'} dz''' \iint dx''' dy''' \frac{1}{\rho A \delta z'} \hat{J}_n(\mathbf{r}''', t) \right] \\
&= \lim_{\substack{\delta z \rightarrow 0 \\ \delta z' \rightarrow 0}} \int_z^{z+\delta z} dz'' \int_{z'}^{z'+\delta z'} dz''' \iiint dx'' dy'' dx''' dy''' \\
&\quad \frac{1}{\rho^2 A^2 \delta z \delta z'} [\hat{J}_n(\mathbf{r}'', t); \hat{J}_m(\mathbf{r}''', t)] \\
&= \lim_{\substack{\delta z \rightarrow 0 \\ \delta z' \rightarrow 0}} \int_z^{z+\delta z} dz'' \int_{z'}^{z'+\delta z'} dz''' \iiint dx'' dy'' dx''' dy''' \\
&\quad \frac{1}{\rho^2 A^2 \delta z \delta z'} \hat{J}_n(\mathbf{r}'', t); i \varepsilon_{nml} \hat{J}_l(\mathbf{r}'') \delta(\mathbf{r}'' - \mathbf{r}'''). \\
&= \lim_{\substack{\delta z \rightarrow 0 \\ \delta z' \rightarrow 0}} \int_z^{z+\delta z} dz'' \int_{z'}^{z'+\delta z'} dz''' \iint dx'' dy'' \\
&\quad \frac{1}{\rho^2 A^2 \delta z \delta z'} \hat{J}_n(\mathbf{r}'', t); i \varepsilon_{nml} \hat{J}_l(\mathbf{r}'') \delta(z'' - z''').
\end{aligned}$$

Here we approximate the function $\delta(z'' - z''')$ that is to be integrated over, by its value at the integration boundary. This only makes sense in the limit of $\delta z \rightarrow 0$, which is the case here. The integration over z''' cancels the term $\delta z'$, and we are left with,

$$\begin{aligned}
&= \frac{i}{\rho A} \delta(z - z') \varepsilon_{nml} \lim_{\delta z \rightarrow 0} \int_z^{z+\delta z} dz'' \iint dx'' dy'' \frac{1}{\rho A \delta z} \hat{J}_l(\mathbf{r}'', t) \\
&= \frac{i}{\rho A} \delta(z - z') \varepsilon_{nml} \hat{J}_l(z, t)
\end{aligned} \tag{B.4.14}$$

To sum up our commutation relation reads:

$$[\hat{J}_n(z, t); \hat{J}_m(z', t)] = \frac{i}{\rho A} \delta(z - z') \varepsilon_{nml} \hat{J}_l(z, t) \tag{B.4.15}$$

B.4.3 Commutation relation on the Local Stokes Operators

The local Stokes operators are defined in equation (2.3.17) Below is shown the commutations relations written in equation (2.3.18). First we consider the x - y commutator.

$$\begin{aligned}
[\hat{S}_x(z, t); \hat{S}_y(z', t)] &= \frac{1}{4} [\hat{a}_1^\dagger(z, t) \hat{a}_1(z, t) - \hat{a}_2^\dagger(z, t) \hat{a}_2(z, t); \hat{a}_1^\dagger(z', t) \hat{a}_2(z', t) + \hat{a}_2^\dagger(z', t) \hat{a}_1(z', t)] \\
&= \frac{1}{4} \{ [\hat{a}_1^\dagger(z, t) \hat{a}_1(z, t); \hat{a}_1^\dagger(z', t) \hat{a}_2(z', t)] + [\hat{a}_1^\dagger(z, t) \hat{a}_1(z, t); \hat{a}_2^\dagger(z', t) \hat{a}_1(z', t)] \\
&\quad - [\hat{a}_2^\dagger(z, t) \hat{a}_2(z, t); \hat{a}_1^\dagger(z', t) \hat{a}_2(z', t)] - [\hat{a}_2^\dagger(z, t) \hat{a}_2(z, t); \hat{a}_2^\dagger(z', t) \hat{a}_1(z', t)] \} \\
&= \frac{1}{4} \{ [\hat{a}_1^\dagger(z, t); \hat{a}_1^\dagger(z', t) \hat{a}_2(z', t)] \hat{a}_1(z, t) + \hat{a}_1^\dagger(z, t) [\hat{a}_1(z, t); \hat{a}_1^\dagger(z', t) \hat{a}_2(z', t)] \\
&\quad + [\hat{a}_1^\dagger(z, t); \hat{a}_2^\dagger(z', t) \hat{a}_1(z', t)] \hat{a}_1(z', t) + \hat{a}_1^\dagger(z, t) [\hat{a}_1(z, t); \hat{a}_2^\dagger(z', t) \hat{a}_1(z', t)] \\
&\quad - [\hat{a}_2^\dagger(z, t); \hat{a}_1^\dagger(z', t) \hat{a}_2(z', t)] \hat{a}_2(z, t) - \hat{a}_2^\dagger(z, t) [\hat{a}_2(z, t); \hat{a}_1^\dagger(z', t) \hat{a}_2(z', t)] \\
&\quad - [\hat{a}_2^\dagger(z, t); \hat{a}_2^\dagger(z', t) \hat{a}_1(z', t)] \hat{a}_2(z, t) - \hat{a}_2^\dagger(z, t) [\hat{a}_2(z, t); \hat{a}_2^\dagger(z', t) \hat{a}_1(z', t)] \}
\end{aligned}$$

Using the commutation relations (2.3.15) we find,

$$\begin{aligned}
&= \frac{1}{4} \{ \hat{a}_1^\dagger(z, t) \hat{a}_2(z, t) - \hat{a}_2^\dagger(z, t) \hat{a}_1(z, t) + \hat{a}_1^\dagger(z, t) \hat{a}_2(z, t) - \hat{a}_2^\dagger(z, t) \hat{a}_1(z, t) \} \delta(z - z') \\
&= i \delta(z - z') \hat{S}_z(z, t)
\end{aligned} \tag{B.4.16}$$

Similarly one may calculate the remaining two commutation relations.

B.4.4 Equation of motion for the Stokes operators

Below is shown the derivation of the modified form of the Heisenberg equation of motion regarding the Stokes operators. One may also read (29) one of the Appendix. Consider the real Heisenberg equation of motion on the annihilation operator given in equation (B.4.6).

$$\frac{\partial}{\partial t} \hat{a}_s(z, t) = \frac{1}{\sqrt{2\pi}} \int \frac{\partial}{\partial t} \hat{a}_s(k, t) e^{ikz} dk = \frac{1}{\sqrt{2\pi}} \int \frac{i}{\hbar} [\mathcal{H}^{\text{Field}} + \mathcal{H}_{\text{EF}}^{\text{Int}}; \hat{a}_s(k, t)] e^{ikz} dk. \tag{B.4.17}$$

The term $[\mathcal{H}^{\text{Field}}; \hat{a}_s(k, t)]$ leads to:

$$[\mathcal{H}^{\text{Field}}; \hat{a}_s(k, t)] = \hbar (\sqrt{\epsilon})^{-1} \omega_k \hat{a}_s(k, t).$$

If one inserts this result, and uses that $\frac{\partial}{\partial z}\hat{a}_s(z, t) = \frac{1}{\sqrt{2\pi}} \int ik\hat{a}_s(k, t)e^{ikz}$, one may find that the equation reduces to:

$$\left(\frac{\partial}{\partial t} + \frac{c}{\sqrt{\epsilon}}\frac{\partial}{\partial z}\right)\hat{a}_s(z, t) = \frac{i}{\hbar}[\mathcal{H}_{\text{Eff}}^{\text{Int}}; \hat{a}_s(k, t)] \quad (\text{B.4.18})$$

B.4.5 Comutation relations, regarding equation (2.3.26)

Introducing the collective operators measuring the collective spin of the ensemble and the number of photons in a laser pulse.

$$\hat{J}_n = \rho A \int_0^L dz \hat{J}_z(z) \quad (\text{B.4.19})$$

$$\hat{S}_n = \frac{c}{\sqrt{\epsilon}} \int_0^T d\tau \hat{S}_n(\tau) \quad (\text{B.4.20})$$

First we look at the commutation relations on the collective spin, these gives:

$$\begin{aligned} [\hat{J}_n; \hat{J}_m] &= \rho^2 A^2 \int_0^L dz \int_0^L dz' [\hat{J}_n(z); \hat{J}_m(z')] \\ &= \rho^2 A^2 \int_0^L dz \int_0^L dz' \frac{i}{\rho A} \epsilon_{nml} \hat{J}_l(z) \delta(z - z') \\ &= i\epsilon_{nml} \rho A \int_0^L dz \hat{J}_l(z) \\ &= i\epsilon_{nml} \hat{J}_l, \end{aligned} \quad (\text{B.4.21})$$

where we used equation (2.3.9). Next we consider the commutation relations on the collective Stokes operators.

$$[\hat{S}_n; \hat{S}_m] = \frac{c^2}{\epsilon} \int_0^T d\tau \int_0^T d\tau' [\hat{S}_n(\tau); \hat{S}_m(\tau')]$$

using that $\tau = t - \frac{\sqrt{\epsilon}}{c}z$ we write:

$$\begin{aligned}
 [\hat{S}_n; \hat{S}_m] &= \frac{\sqrt{\epsilon}}{c} \int_0^T d\tau \int_0^L dz [\hat{S}_n(z, t); \hat{S}_m(z', t)] \\
 &= \frac{\sqrt{\epsilon}}{c} \int_0^T d\tau \int_0^L dz i\epsilon_{nml} \hat{S}_l(z, t) \delta(z - z') \\
 &= i\epsilon_{nml} \frac{\sqrt{\epsilon}}{c} \int_0^T d\tau \hat{S}_l(\tau) \\
 &= i\epsilon_{nml} \hat{S}_l
 \end{aligned} \tag{B.4.22}$$

Detailed calculations, Part II

C.1 Deriving Euler Lagrange Equations of Motion from Hamiltons Principle

The derivation follows the one described in (44). The Lagrangian density $\mathcal{L}(\phi_r, \partial_\mu \phi_r)$, where ϕ_r denote a set of fields, is related to the the action $S(\phi_r)$ as.

$$S(\phi_r) = \int d^n x \mathcal{L}(\phi_r, \partial_\mu \phi_r). \quad (\text{C.1.1})$$

A variation if the action denoted $\delta S(\phi_r)$ corresponding to fixed boundary conditions can the be found to give:

$$\delta S(\phi_r) = \int d^n x \left\{ \frac{\mathcal{L}}{\partial \phi_r} \delta \phi_r(\mathbf{x}) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_r)} \delta \partial_\mu \phi_r(\mathbf{x}) \right\}. \quad (\text{C.1.2})$$

Using the following calculation,

$$\delta \partial_\mu \phi_r(\mathbf{x}) = \partial_\mu \{ \phi_r(\mathbf{x}) + \delta \phi_r(\mathbf{x}) \} - \partial_\mu \phi_r(\mathbf{x}) = \partial_\mu \delta \phi_r(\mathbf{x}) \quad (\text{C.1.3})$$

we may write:

$$= \int d^n x \left\{ \frac{\mathcal{L}}{\partial \phi_r} \delta \phi_r(\mathbf{x}) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_r)} \partial_\mu \delta \phi_r(\mathbf{x}) \right\}. \quad (\text{C.1.4})$$

Assuming fixed boundary conditions such that the variation $\delta\phi_r(\mathbf{x})$ vanish at the boundary, we may use partial integration once, and find:

$$= \int d^n x \left\{ \frac{\mathcal{L}}{\partial\phi_r} \delta\phi_r(\mathbf{x}) - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)} \delta\phi_r(\mathbf{x}) \right\} = \int d^n x \left\{ \frac{\mathcal{L}}{\partial\phi_r} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)} \right\} \delta\phi_r(\mathbf{x}). \quad (\text{C.1.5})$$

Since the variation of the action has to vanish for every field variation $\delta\phi_r(\mathbf{x})$ we derive the Euler Lagrange Equations of motion:

$$\frac{\mathcal{L}}{\partial\phi_r} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)} = 0 \quad (\text{C.1.6})$$

The important thing to notice here is that the Euler Lagrange equations of motion stems from an integral equations, and manipulations such as partial integrations does not change the validity of the equations of motion.

C.2 Lagrange Equations

Lorentz Force

Doing the functional derivatives on the Lagrangian density with respect to the field $\chi(\mathbf{r}, t)$ we find:

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\chi_r} &= -e \frac{\partial U_{\text{Atom}}(\chi)}{\partial\chi_r} + g \left(-\frac{\partial A_r(\mathbf{r}, t)}{\partial t} - (\nabla\phi(\mathbf{r}, t))_r \right) \\ &= -e \frac{\partial U_{\text{Atom}}(\chi)}{\partial\chi_r} + gE_r(\mathbf{r}, t) \\ \frac{\partial}{\partial t} \frac{\partial\mathcal{L}}{\partial\dot{\chi}_r} &= m \frac{\partial^2}{\partial t^2} \chi_r(\mathbf{r}, t) \end{aligned}$$

on vector form this reads,

$$m \frac{\partial^2}{\partial t^2} \chi(\mathbf{r}, t) = -e \frac{\partial U_{\text{Atom}}(\chi)}{\partial\chi} + g\mathbf{E}(\mathbf{r}, t) \quad (\text{C.2.1})$$

Maxwell Equations

Let us consider the equations one may derive from Euler Lagrange equation of motion, considering the vector potential $\mathbf{A}(\mathbf{r}, t)$.

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial A_r} &= \frac{-1}{2\mu_0} \left(\frac{\partial(\nabla \times \mathbf{A})_k}{\partial A_r} (\nabla \times \mathbf{A})_k + (\nabla \times \mathbf{A})_k \frac{\partial(\nabla \times \mathbf{A})_k}{\partial A_r} \right) \\ &= \frac{-1}{\mu_0} (\nabla \times \mathbf{A})_k \varepsilon_{knm} \frac{\partial}{\partial r_n} \frac{\partial A_m}{\partial A_r}.\end{aligned}$$

Imagining we were using the Euler Lagrange equations of motion on integral form*, leaves us in a position where we may use partial integration. To shorten notation we will not write the integration, only remind ourselves of this connection. Partial integration thus gives:

$$\begin{aligned}&= \frac{1}{\mu_0} \varepsilon_{knm} \frac{\partial}{\partial r_n} (\nabla \times \mathbf{A})_k \delta_{mr} \\ &= \frac{-1}{\mu_0} \varepsilon_{rnk} \frac{\partial}{\partial r_n} (\nabla \times \mathbf{A})_k \\ &= \frac{-1}{\mu_0} (\nabla \times \nabla \times \mathbf{A})_r \\ &= -\frac{1}{\mu_0} (\nabla \times \mathbf{B}(\mathbf{r}, t))_r.\end{aligned}\tag{C.2.2a}$$

Similarly we calculate the remaining term,

$$\begin{aligned}\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial (\frac{\partial}{\partial t} A_r)} &= \frac{\partial}{\partial t} \left\{ \varepsilon_0 \left(\frac{\partial}{\partial t} A_r + (\nabla \phi)_r \right) + e\chi_r \right\} \\ &= -\frac{\partial}{\partial t} \left\{ \varepsilon_0 E_r - e\chi_r \right\} \\ &\equiv -\frac{\partial}{\partial t} D_r(\mathbf{r}, t).\end{aligned}\tag{C.2.2b}$$

Combining the two equations (C.2.2a) and (C.2.2b) we find the first Maxwell equation.

$$\frac{\partial}{\partial t} \mathbf{D}(\mathbf{r}, t) = \frac{1}{\mu_0} \nabla \times \mathbf{B}(\mathbf{r}, t)\tag{C.2.3}$$

Similarly we may calculate for the remaining term:

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial (\frac{\partial \phi}{\partial t})} = 0.\tag{C.2.4a}$$

The other term we calculate as,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \phi} &= \epsilon_0 \left(-\frac{\partial}{\partial t} A_n - (\nabla \phi)_n \right) \frac{\partial}{\partial \phi} \left(-\frac{\partial}{\partial t} A_n - (\nabla \phi)_n \right) + e \frac{\partial}{\partial \phi} (\nabla \phi)_n \chi_n \\ &= -\epsilon_0 \left(-\frac{\partial}{\partial t} A_n - (\nabla \phi)_n \right) \frac{\partial}{\partial \phi} (\nabla \phi)_n + e \frac{\partial}{\partial \phi} (\nabla \phi)_n \chi_n.\end{aligned}$$

Again we use partial integration on both terms and find,

$$\begin{aligned}&= \epsilon_0 \nabla \cdot \left(-\frac{\partial}{\partial t} \mathbf{A} - (\nabla \phi) \right) \frac{\partial \phi}{\partial \phi} + \nabla \cdot e \chi \frac{\partial \phi}{\partial \phi} \\ &= \epsilon_0 \nabla \cdot \left\{ \left(-\frac{\partial}{\partial t} \mathbf{A} - (\nabla \phi) \right) + e \chi \right\} \\ &= \nabla \cdot \mathbf{D}(\mathbf{r}, t).\end{aligned}\tag{C.2.4b}$$

Combining the two equations (C.2.4a) and (C.2.4b) we find the final Maxwell equation.

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = 0\tag{C.2.5}$$

C.3 Remarks on Legendre transformation

First we calculate the canonical momentas.

$$\Pi_r = \frac{\partial \mathcal{L}}{\partial \dot{A}_r} = \epsilon_0 \left(\frac{\partial A_r}{\partial t} + (\nabla \phi)_r \right) + e \chi_r = -D_r(\mathbf{r}, t)\tag{C.3.1a}$$

or

$$\frac{\partial A_r}{\partial t} = \frac{\Pi_r}{\epsilon_0} - (\nabla \phi)_r - \frac{e}{\epsilon_0} \chi_r.$$

Similarly we find

$$\Phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0\tag{C.3.1b}$$

and

$$\theta_r = \frac{\partial \mathcal{L}}{\partial \dot{\chi}_r} = m \frac{\partial \chi_r}{\partial t}\tag{C.3.1c}$$

or

$$\frac{\partial \chi_r}{\partial t} = \frac{\theta_r}{m}.$$

These identities we use in equation (4.1.1) along with the Lagrangian (3.1.6), and derive our Hamiltonian. To shorten notation we suppress time- and space dependency.

$$\begin{aligned} \mathcal{H} &= \int d^3 r \left(\frac{\mathbf{\Pi}}{\epsilon_0} - \nabla \phi - \frac{e}{\epsilon_0} \boldsymbol{\chi} \right) \cdot \mathbf{\Pi} + \frac{\boldsymbol{\theta}^2}{m} - \frac{1}{2} \epsilon_0 \left(-\frac{\mathbf{\Pi}}{\epsilon_0} + \nabla \phi + \frac{e}{\epsilon_0} \boldsymbol{\chi} - \nabla \phi \right)^2 \\ &\quad + \frac{1}{2} \frac{1}{\mu_0} (\nabla \times \mathbf{A})^2 + e \left(-\frac{\mathbf{\Pi}}{\epsilon_0} + \nabla \phi + \frac{e}{\epsilon_0} \boldsymbol{\chi} - \nabla \phi \right) \cdot \boldsymbol{\chi} \\ &\quad - \frac{\boldsymbol{\theta}^2}{2m} + eU_{\text{atom}}(\boldsymbol{\chi}) \\ &= \int d^3 r \left(\frac{\mathbf{\Pi}}{\epsilon_0} - \nabla \phi - \frac{e}{\epsilon_0} \boldsymbol{\chi} \right) \cdot \mathbf{\Pi} + \frac{\boldsymbol{\theta}^2}{m} - \frac{1}{2} \epsilon_0 \left(-\frac{\mathbf{\Pi}}{\epsilon_0} + \frac{e}{\epsilon_0} \boldsymbol{\chi} \right)^2 \\ &\quad + \frac{1}{2} \frac{1}{\mu_0} (\nabla \times \mathbf{A})^2 + e \left(-\frac{\mathbf{\Pi}}{\epsilon_0} + \frac{e}{\epsilon_0} \boldsymbol{\chi} \right) \cdot \boldsymbol{\chi} \\ &\quad - \frac{\boldsymbol{\theta}^2}{2m} + eU_{\text{atom}}(\boldsymbol{\chi}) \\ &= \int d^3 r \frac{\mathbf{\Pi}^2}{2\epsilon_0} + \frac{(\nabla \times \mathbf{A})^2}{2\mu_0} + \frac{e^2 \boldsymbol{\chi}^2}{2\epsilon_0 m} - \nabla \phi \cdot \mathbf{\Pi} - \frac{e}{\epsilon_0} \mathbf{\Pi} \cdot \boldsymbol{\chi} \\ &\quad + \frac{\boldsymbol{\theta}^2}{2m} + eU_{\text{atom}}(\boldsymbol{\chi}). \end{aligned}$$

Hence our classical Hamiltonian is given by

$$\mathcal{H} = \int d^3 r \frac{1}{2} \left(\frac{\mathbf{\Pi}^2}{\epsilon_0} + \frac{(\nabla \times \mathbf{A})^2}{\mu_0} + \frac{e^2 \boldsymbol{\chi}^2}{\epsilon_0 m} \right) - \nabla \phi \cdot \mathbf{\Pi} - \frac{e}{\epsilon_0} \mathbf{\Pi} \cdot \boldsymbol{\chi} + \frac{\boldsymbol{\theta}^2}{2m} + eU_{\text{atom}}(\boldsymbol{\chi}). \quad (\text{C.3.2})$$

C.4 Maxwell Equations derived from Hamilton's equations of motion

Using equation (4.1.4) and equation (4.1.3) we may calculate the following.

$$\begin{aligned}
 \frac{\partial(\nabla \times \mathbf{A})_r}{\partial t} &= \left(\nabla \times \frac{\partial \mathbf{A}}{\partial t} \right)_r = \varepsilon_{rjk} \frac{\partial}{\partial r_j} \frac{\partial A_k}{\partial t} = \varepsilon_{rjk} \frac{\partial}{\partial r_j} \frac{\partial \mathcal{H}}{\partial \Pi_k} \\
 &= \varepsilon_{rjk} \frac{\partial}{\partial r_j} \left(\frac{\Pi_k}{\epsilon_0} - (\nabla \phi)_k - \frac{e}{\epsilon_0} \chi_k \right) \\
 &= \varepsilon_{rjk} \frac{\partial}{\partial r_j} \left(\frac{\partial A_k}{\partial t} \right) + (\nabla \times \nabla \phi)_r \\
 &= -(\nabla \times \mathbf{E}(\mathbf{r}, t))_r
 \end{aligned}$$

This is the first Maxwell equation.

$$\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} = -\nabla \times \mathbf{E}(\mathbf{r}, t) \quad (\text{C.4.1})$$

In order to calculate the second Maxwell equation we have to use partial integration, following same reasoning as in equation (C.2.2a).

$$\begin{aligned}
 \int d^3 r \frac{\partial \Pi_r}{\partial t} &= - \int d^3 r \frac{\partial \mathcal{H}}{\partial A_r} = -\frac{1}{\mu_0} \int d^3 r \varepsilon_{mjk} (\nabla \times \mathbf{A})_m \frac{\partial}{\partial r_j} \frac{\partial A_k}{\partial A_r} \\
 &= - \int d^3 r \frac{1}{\mu_0} \varepsilon_{rjm} \frac{\partial}{\partial r_j} (\nabla \times \mathbf{A})_m \\
 &= - \int d^3 r \frac{1}{\mu_0} (\nabla \times \nabla \times \mathbf{A})_r
 \end{aligned}$$

Hence we found the Maxwell equation

$$\frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} = \frac{1}{\mu_0} \nabla \times \mathbf{B}(\mathbf{r}, t) \quad (\text{C.4.2})$$

Again we use partial integration, and find the last Maxwell equation,

$$\begin{aligned}
 0 &= \int d^3 r \frac{\partial \theta_r}{\partial t} = - \int d^3 r \frac{\partial \mathcal{H}}{\partial \phi_r} \\
 &= \int d^3 r \frac{\partial \Pi_k \nabla \phi_k}{\partial \phi_r} = - \int d^3 r (\nabla \cdot \mathbf{\Pi})_r
 \end{aligned}$$

that is

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = 0. \quad (\text{C.4.3})$$

C.5 Remarks on the basis $\{\mathbf{g}_k(\mathbf{r})\}$

We define the inner product to be the usual,

$$\langle \boldsymbol{\phi}(\mathbf{r}, t) | \boldsymbol{\psi}(\mathbf{r}, t) \rangle = \int d^3r \boldsymbol{\phi}(\mathbf{r}, t) \cdot \boldsymbol{\psi}^*(\mathbf{r}, t). \quad (\text{C.5.1})$$

We then calculate the inner product,

$$\begin{aligned} \langle \boldsymbol{\phi}(\mathbf{r}, t) | D[\boldsymbol{\psi}(\mathbf{r}, t)] \rangle &= \langle \boldsymbol{\phi}(\mathbf{r}, t) | \frac{1}{\sqrt{\epsilon(\mathbf{r})}} \nabla \times \nabla \times \frac{1}{\sqrt{\epsilon(\mathbf{r})}} \boldsymbol{\psi}(\mathbf{r}, t) \rangle \\ &= \int d^3r \boldsymbol{\phi}(\mathbf{r}, t) \cdot \frac{1}{\sqrt{\epsilon(\mathbf{r})}} \nabla \times \nabla \times \frac{1}{\sqrt{\epsilon(\mathbf{r})}} \boldsymbol{\psi}^*(\mathbf{r}, t) \\ &= \int d^3r \frac{1}{\sqrt{\epsilon(\mathbf{r})}} \boldsymbol{\phi}(\mathbf{r}, t) \cdot \nabla \times \nabla \times \frac{1}{\sqrt{\epsilon(\mathbf{r})}} \boldsymbol{\psi}^*(\mathbf{r}, t) \end{aligned}$$

using partial integration twice we find the following

$$\begin{aligned} &= \int d^3r \nabla \times \nabla \times \frac{1}{\sqrt{\epsilon(\mathbf{r})}} \boldsymbol{\phi}(\mathbf{r}, t) \cdot \frac{1}{\sqrt{\epsilon(\mathbf{r})}} \boldsymbol{\psi}^*(\mathbf{r}, t) \\ &= \int d^3r \frac{1}{\sqrt{\epsilon(\mathbf{r})}} \nabla \times \nabla \times \frac{1}{\sqrt{\epsilon(\mathbf{r})}} \boldsymbol{\phi}(\mathbf{r}, t) \cdot \boldsymbol{\psi}^*(\mathbf{r}, t) \\ &= \langle \frac{1}{\sqrt{\epsilon(\mathbf{r})}} \nabla \times \nabla \times \frac{1}{\sqrt{\epsilon(\mathbf{r})}} \boldsymbol{\phi}(\mathbf{r}, t) | \boldsymbol{\psi}(\mathbf{r}, t) \rangle. \end{aligned} \quad (\text{C.5.2})$$

Thus we conclude that the differential operator $D[\cdot]$ is hermitian and the set $\{\mathbf{g}_k(\mathbf{r})\}$ may be chosen as a basis for the space of L_2 functions subject to requirement (4.2.11b).

Remarks on the Hamiltonian equation (4.2.27)

In the following we bring the Hamiltonian on quadratic form.

$$\mathcal{H} = \int d^3r \frac{1}{2} \left\{ \frac{1}{\epsilon_0 \epsilon(\mathbf{r})} \boldsymbol{\Pi}(\mathbf{r}, t)^2 + \frac{1}{\mu_0} (\nabla \times \mathbf{A}(\mathbf{r}, t))^2 \right\}.$$

Using partial integration assuming appropriate boundary conditions on the field $\mathbf{A}(\mathbf{r}, t)$, we write

$$= \int d^3r \frac{1}{2} \left\{ \frac{1}{\epsilon_0 \epsilon(\mathbf{r})} \boldsymbol{\Pi}(\mathbf{r}, t)^2 + \frac{1}{\mu_0} (\mathbf{A}(\mathbf{r}, t) \cdot \nabla \times \nabla \times \mathbf{A}(\mathbf{r}, t)) \right\}.$$

Inserting the expansions (4.2.20) and (4.2.21) into the Hamiltonian (4.2.5) we find:

$$= \sum_{\mathbf{k}\mathbf{k}'} \int d^3r \frac{1}{2} \left\{ \frac{1}{\epsilon(\mathbf{r})} \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \cdot \mathbf{f}_{\mathbf{k}'}^*(\mathbf{r}) P_{\mathbf{k}'}^\dagger(t) P_{\mathbf{k}}(t) + \omega_{\mathbf{k}'}^2 \frac{1}{\epsilon(\mathbf{r})} \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \cdot \mathbf{f}_{\mathbf{k}'}^*(\mathbf{r}) Q_{\mathbf{k}'}^\dagger(t) Q_{\mathbf{k}}(t) \right\}.$$

We then use equation (4.2.16)

$$\begin{aligned} &= \sum_{\mathbf{k}\mathbf{k}'} \frac{1}{2} \left\{ P_{\mathbf{k}'}^\dagger(t) P_{\mathbf{k}}(t) \delta_{\mathbf{k}\mathbf{k}'} + \omega_{\mathbf{k}'}^2 Q_{\mathbf{k}'}^\dagger(t) Q_{\mathbf{k}}(t) \delta_{\mathbf{k}\mathbf{k}'} \right\} \\ \mathcal{H} &= \frac{1}{2} \left\{ P_{\mathbf{k}}^\dagger(t) P_{\mathbf{k}}(t) + \omega_{\mathbf{k}}^2 Q_{\mathbf{k}}^\dagger(t) Q_{\mathbf{k}}(t) \right\} \end{aligned} \quad (\text{C.5.3})$$

Remarks on the unitary operator U

In the following we show that the operator U is unitary.

$$\begin{aligned} \sum_{\mathbf{k}'} U_{\mathbf{k}\mathbf{k}'} U_{\mathbf{k}''\mathbf{k}'}^* &= \sum_{\mathbf{k}'} \int d^3r \frac{1}{\epsilon(\mathbf{r})} \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \cdot \mathbf{f}_{\mathbf{k}'}(\mathbf{r}) \int d^3r' \frac{1}{\epsilon(\mathbf{r}')} \mathbf{f}_{\mathbf{k}''}^*(\mathbf{r}') \cdot \mathbf{f}_{\mathbf{k}'}^*(\mathbf{r}') \\ &= \iint d^3r d^3r' \frac{1}{\epsilon(\mathbf{r})} \frac{1}{\epsilon(\mathbf{r}')} \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \cdot \sum_{\mathbf{k}'} \mathbf{f}_{\mathbf{k}'}(\mathbf{r}) \mathbf{f}_{\mathbf{k}'}^*(\mathbf{r}') \cdot \mathbf{f}_{\mathbf{k}}^*(\mathbf{r}') \\ &= \iint d^3r d^3r' \frac{1}{\epsilon(\mathbf{r})} \frac{1}{\epsilon(\mathbf{r}')} \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \cdot \bar{\delta}^{\mathbf{T}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{f}_{\mathbf{k}}^*(\mathbf{r}') \end{aligned}$$

Using equation (4.2.19) we find

$$\begin{aligned} &= \int d^3r \frac{1}{\epsilon(\mathbf{r})} \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \cdot \mathbf{f}_{\mathbf{k}}^*(\mathbf{r}) \\ &= \delta_{\mathbf{k}\mathbf{k}'} \end{aligned} \quad (\text{C.5.4})$$

Remarks on the Hamiltonian equation (4.2.33)

$$\begin{aligned}
\mathcal{H} &= \sum_{\mathbf{k}} \frac{1}{2} \{P_{\mathbf{k}}^* P_{\mathbf{k}} + \omega_{\mathbf{k}}^2 Q_{\mathbf{k}}^* Q_{\mathbf{k}}\} \\
&= \frac{1}{2} \sum_{\mathbf{k}} \left\{ -i \sqrt{\frac{\hbar \omega_{\mathbf{k}}}{2}} \left\{ \hat{a}_{\mathbf{k}} - \sum_{\mathbf{k}'} U_{\mathbf{k}\mathbf{k}'}^* \hat{a}_{\mathbf{k}'}^\dagger \right\} i \sqrt{\frac{\hbar \omega_{\mathbf{k}}}{2}} \left\{ \hat{a}_{\mathbf{k}}^\dagger - \sum_{\mathbf{k}'} U_{\mathbf{k}\mathbf{k}'} \hat{a}_{\mathbf{k}'} \right\} \right. \\
&\quad \left. + \omega_{\mathbf{k}}^2 \sqrt{\frac{\hbar}{2\omega_{\mathbf{k}}}} \left\{ \hat{a}_{\mathbf{k}}^\dagger + \sum_{\mathbf{k}''} U_{\mathbf{k}\mathbf{k}''} \hat{a}_{\mathbf{k}''} \right\} \sqrt{\frac{\hbar}{2\omega_{\mathbf{k}}}} \left\{ \hat{a}_{\mathbf{k}} + \sum_{\mathbf{k}''} U_{\mathbf{k}\mathbf{k}''}^* \hat{a}_{\mathbf{k}''}^\dagger \right\} \right\} \\
&= \frac{1}{2} \sum_{\mathbf{k}} \hbar \omega_{\mathbf{k}} \frac{1}{2} \left\{ \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}}^\dagger + \underbrace{\sum_{\mathbf{k}'\mathbf{k}''} U_{\mathbf{k}\mathbf{k}'}^* U_{\mathbf{k}\mathbf{k}''} \hat{a}_{\mathbf{k}'}^\dagger \hat{a}_{\mathbf{k}''}}_{*} + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \underbrace{\sum_{\mathbf{k}'\mathbf{k}''} U_{\mathbf{k}\mathbf{k}'} U_{\mathbf{k}\mathbf{k}''}^* \hat{a}_{\mathbf{k}'} \hat{a}_{\mathbf{k}''}^\dagger}_{**} \right\} \quad (\text{C.5.5})
\end{aligned}$$

The mixed terms cancel, and what is left to consider is the type of term marked by * and **. Let us consider the term *, using the short notation for normalized inner product on the basis $\{\mathbf{f}_k(\mathbf{r})\}$

$$\begin{aligned}
\sum_{\mathbf{k}'\mathbf{k}''} U_{\mathbf{k}\mathbf{k}'}^* U_{\mathbf{k}\mathbf{k}''} \hat{a}_{\mathbf{k}'}^\dagger \hat{a}_{\mathbf{k}''} &= \sum_{\mathbf{k}'\mathbf{k}''} \langle \mathbf{f}_{\mathbf{k}}^* | \mathbf{f}_{\mathbf{k}'} \rangle \langle \mathbf{f}_{\mathbf{k}} | \mathbf{f}_{\mathbf{k}''} \rangle \hat{a}_{\mathbf{k}'}^\dagger \hat{a}_{\mathbf{k}''} \\
&= \sum_{\substack{\mathbf{k}^{(1)}\mathbf{k}^{(2)} \\ \mathbf{k}^{(3)}\mathbf{k}^{(4)}}} \langle \mathbf{f}_{\mathbf{k}^{(1)}}^* | \mathbf{f}_{\mathbf{k}^{(3)}} \rangle \langle \mathbf{f}_{\mathbf{k}^{(3)}} | \mathbf{f}_{\mathbf{k}^{(2)}} \rangle \langle \mathbf{f}_{\mathbf{k}^{(2)}} | \mathbf{f}_{\mathbf{k}^{(4)}} \rangle \langle \mathbf{f}_{\mathbf{k}^{(4)}}^* | \mathbf{f}_{\mathbf{k}^{(1)}} \rangle \hat{a}_{\mathbf{k}^{(1)}}^\dagger \hat{a}_{\mathbf{k}^{(2)}} \quad (\text{C.5.6})
\end{aligned}$$

The meaning of equation (C.5.6), is shown in the diagram in Figure C.1. Since inner product is

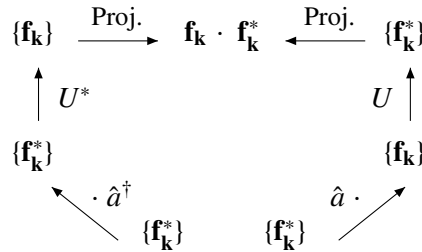


Figure C.1: Diagrammatic interpretation of equation (C.5.6)

conserved under the unitary operator U the above is equivalent to:

$$\int d^3 r \hat{a}_{\mathbf{k}}^\dagger \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \cdot \hat{a}_{\mathbf{k}} \mathbf{f}_{\mathbf{k}}^* \quad (\text{C.5.7})$$

We conclude that $*$ is equivalent to the following

$$\sum_{\mathbf{k}'\mathbf{k}''} U_{\mathbf{k}\mathbf{k}'}^* U_{\mathbf{k}\mathbf{k}''} \hat{a}_{\mathbf{k}'}^\dagger \hat{a}_{\mathbf{k}''} = \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} \quad (\text{C.5.8})$$

Similarly the term $**$ may be calculated, and the Hamiltonian (C.5.5) reduce to equation (4.2.33).

C.5.1 Remarks on expansion of $\hat{\mathbf{A}}(\mathbf{r}, t)$ equation (4.2.34)

We find the field $\hat{\mathbf{A}}(\mathbf{r}, t)$ given in terms of creation and annihilation operators by inserting equation (4.2.31) into expansion (4.2.21).

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) &= \sum_{\mathbf{k}} c \sqrt{\mu_0} Q_{\mathbf{k}}(t) \frac{\mathbf{f}_{\mathbf{k}}(\mathbf{r})}{\epsilon(\mathbf{r})} \\ &= \sum_{\mathbf{k}} \frac{c}{\epsilon(\mathbf{r})} \sqrt{\frac{\hbar\mu_0}{2\omega_{\mathbf{k}}}} \left\{ \hat{a}_{\mathbf{k}}(t) \mathbf{f}_{\mathbf{k}}(\mathbf{r}) + \sum_{\mathbf{k}'} U_{\mathbf{k}\mathbf{k}'}^* \hat{a}_{\mathbf{k}'}^\dagger(t) \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \right\} \\ &= \sum_{\mathbf{k}} \frac{c}{\epsilon(\mathbf{r})} \sqrt{\frac{\hbar\mu_0}{2\omega_{\mathbf{k}}}} \left\{ \hat{a}_{\mathbf{k}}(t) \mathbf{f}_{\mathbf{k}}(\mathbf{r}) + \underbrace{\sum_{\mathbf{k}'\mathbf{k}''} \hat{a}_{\mathbf{k}'}^\dagger(t) U_{\mathbf{k}\mathbf{k}'}^* U_{\mathbf{k}\mathbf{k}''} \mathbf{f}_{\mathbf{k}''}^*(\mathbf{r})}_{\ddagger} \right\} \end{aligned} \quad (\text{C.5.9})$$

A diagrammatic interpretation of the term \ddagger is depicted in Figure C.2. We see that since the

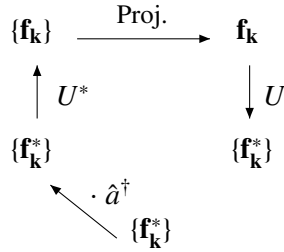


Figure C.2: Diagrammatic interpretation of the term \ddagger in equation C.5.9

transformation U is unitary the bijective translation of the basis $\cdot \hat{a}^\dagger$ is conserved and the term \ddagger is equivalent to:

$$\hat{a}_{\mathbf{k}}^\dagger(t) \mathbf{f}_{\mathbf{k}}^*(\mathbf{r}) \quad (\text{C.5.10})$$

The quantized vector potential expanded in creation and annihilation operators may thus be written as

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}} \frac{c}{\epsilon(\mathbf{r})} \sqrt{\frac{\hbar\mu_0}{2\omega_{\mathbf{k}}}} \{ \hat{a}_{\mathbf{k}}(t) \mathbf{f}_{\mathbf{k}}(\mathbf{r}) + \hat{a}_{\mathbf{k}}^\dagger(t) \mathbf{f}_{\mathbf{k}}^*(\mathbf{r}) \} \quad (\text{C.5.11})$$

C.6 Deriving Maxwell equations

C.6.1 Remarks on Maxwell equation (5.1.1)

In the following we derive the first Maxwell equation, using *Heisenberg equation of motion*. We do our calculations on each component of the vector.

$$\begin{aligned} \frac{\partial}{\partial t} \hat{D}_r(\mathbf{r}, t) &= \frac{i}{\hbar} [\hat{\mathcal{H}}_{\text{Eff}}; \hat{D}_r(\mathbf{r}, t)] \\ &= \frac{i}{\hbar} \int d^3 r' \frac{1}{2\mu_0} [(\nabla \times \hat{\mathbf{A}}(\mathbf{r}', t))^2; \hat{D}_r(\mathbf{r}, t)] \\ &= \frac{i}{\hbar} \int d^3 r' \frac{1}{\mu_0} (\nabla \times \hat{\mathbf{A}}(\mathbf{r}', t))_n \epsilon_{nml} \frac{\partial}{\partial r'_m} [\hat{A}_l(\mathbf{r}', t); \hat{D}_r(\mathbf{r}, t)]. \end{aligned}$$

Using partial integration once we find,

$$\begin{aligned} \frac{\partial}{\partial t} \hat{D}_r(\mathbf{r}, t) &= \frac{i}{\hbar} \int d^3 r' \frac{1}{\mu_0} \epsilon_{lmn} \frac{\partial}{\partial r'_m} (\nabla \times \hat{\mathbf{A}}(\mathbf{r}', t))_n [\hat{A}_l(\mathbf{r}', t); \hat{D}_r(\mathbf{r}, t)] \\ &= - \frac{i}{\hbar} \int d^3 r' \frac{1}{\mu_0} (\nabla \times \nabla \times \hat{\mathbf{A}}(\mathbf{r}', t))_l [\hat{A}_l(\mathbf{r}', t); \hat{D}_r(\mathbf{r}, t)]. \end{aligned}$$

Using commutation relation (4.3.2), on component form, we find:

$$\frac{\partial}{\partial t} \hat{D}_r(\mathbf{r}, t) = - \frac{i}{\hbar} \int d^3 r' \frac{1}{\mu_0} (\nabla \times \nabla \times \hat{\mathbf{A}}(\mathbf{r}', t))_l i\hbar \delta_{lr}^T(\mathbf{r}', \mathbf{r})$$

Since the field $\nabla \times \nabla \times \hat{\mathbf{A}}(\mathbf{r}', t)$ is transverse, we use identity (4.2.19) and find,

$$\frac{\partial}{\partial t} \hat{D}_r(\mathbf{r}, t) = (\nabla \times \nabla \times \hat{\mathbf{A}}(\mathbf{r}, t))_r \quad (\text{C.6.1})$$

This we may write on vector form, and use the definition of the magnetic field (3.1.5b)

$$\frac{\partial}{\partial t} \hat{\mathbf{D}}(\mathbf{r}, t) = \frac{1}{\mu_0} \nabla \times \hat{\mathbf{B}}(\mathbf{r}, t). \quad (\text{C.6.2})$$

This is one of the Maxwell equations, also seen in equation (4.2.7a).

C.6.2 Remarks on Maxwell equation (5.1.2)

Below we calculate the second Maxwell equation.

$$\begin{aligned} \frac{\partial}{\partial t} (\nabla \times \hat{\mathbf{A}}(\mathbf{r}, t))_r &= \varepsilon_{rnm} \frac{\partial}{\partial r_n} \frac{\partial \hat{A}_m(\mathbf{r}, t)}{\partial t} \\ &= \varepsilon_{rnm} \frac{\partial}{\partial r_n} \frac{i}{\hbar} [\hat{\mathcal{H}}_{\text{eff}}; \hat{A}_m(\mathbf{r}, t)] \\ &= \varepsilon_{rnm} \frac{\partial}{\partial r_n} \frac{i}{\hbar} \left\{ \int d^3 r' \frac{1}{2\epsilon_0} [\hat{\mathbf{D}}(\mathbf{r}', t)^2; \hat{A}_m(\mathbf{r}, t)] \right. \\ &\quad \left. - \sum_j^{\text{atoms}} \frac{2\beta}{\epsilon_0} [\hat{\mathbf{D}}^+(\mathbf{r}_j, t) \bar{\alpha}_j \hat{\mathbf{D}}^-(\mathbf{r}_j, t); \hat{A}_m(\mathbf{r}, t)] \right\}. \end{aligned}$$

The first commutator $[[\hat{\mathbf{D}}(\mathbf{r}', t)^2; \hat{A}_m(\mathbf{r}, t)]$ we may write:

$$[\hat{\mathbf{D}}(\mathbf{r}', t)^2; \hat{A}_m(\mathbf{r}, t)] = \hat{D}_l(\mathbf{r}', t) [\hat{D}_l(\mathbf{r}', t); \hat{A}_m(\mathbf{r}, t)] + [\hat{D}_l(\mathbf{r}', t); \hat{A}_m(\mathbf{r}, t)] \hat{D}_l(\mathbf{r}', t).$$

Using commutation relation (4.3.2) and integrating this term out we find:

$$\begin{aligned} \frac{\partial}{\partial t} (\nabla \times \hat{\mathbf{A}}(\mathbf{r}, t))_r &= -\frac{1}{\epsilon_0} (\nabla \times \hat{\mathbf{D}}(\mathbf{r}, t))_r - \varepsilon_{rnm} \frac{\partial}{\partial r_n} \frac{i}{\hbar} \sum_j^{\text{atoms}} \frac{2\beta}{\epsilon_0} [\hat{\mathbf{D}}^+(\mathbf{r}_j, t) \bar{\alpha}_j \hat{\mathbf{D}}^-(\mathbf{r}_j, t); \hat{A}_m(\mathbf{r}, t)] \\ &= -\frac{1}{\epsilon_0} (\nabla \times \hat{\mathbf{D}}(\mathbf{r}, t))_r - \varepsilon_{rnm} \frac{\partial}{\partial r_n} \frac{i}{\hbar} \sum_j^{\text{atoms}} \frac{2\beta}{\epsilon_0} \left\{ [\hat{D}_s^+(\mathbf{r}_j, t); \hat{A}_m(\mathbf{r}, t)] (\bar{\alpha}_j \hat{\mathbf{D}}^-(\mathbf{r}_j, t))_s \right. \\ &\quad \left. + \hat{D}_s^+(\mathbf{r}_j, t) [(\bar{\alpha}_j \hat{\mathbf{D}}^-(\mathbf{r}_j, t))_s; \hat{A}_m(\mathbf{r}, t)] \right\}. \end{aligned}$$

Using relation (4.2.51b), leads to:

$$\begin{aligned} \frac{\partial}{\partial t} (\nabla \times \hat{\mathbf{A}}(\mathbf{r}, t))_r &= -\frac{1}{\epsilon_0} (\nabla \times \hat{\mathbf{D}}(\mathbf{r}, t))_r - \epsilon_{rnm} \frac{\partial}{\partial r_n} \frac{i}{\hbar} \sum_j^{\text{atoms}} \frac{2\beta}{\epsilon_0} \left\{ \frac{i\hbar}{2} \delta_{ms}^T(\mathbf{r}, \mathbf{r}_j) (\bar{\alpha}_j \hat{\mathbf{D}}^-(\mathbf{r}_j, t))_s \right. \\ &\quad \left. + \hat{D}_s^+(\mathbf{r}_j, t) [(\bar{\alpha}_j \hat{\mathbf{D}}^-(\mathbf{r}_j, t))_s; \hat{A}_m(\mathbf{r}, t)] \right\}. \end{aligned} \quad (\text{C.6.3})$$

Let us then calculate the last term of equation (C.6.3)

$$\begin{aligned} \hat{D}_s^+(\mathbf{r}_j, t) [(\bar{\alpha}_j \hat{\mathbf{D}}^-(\mathbf{r}_j, t))_s; \hat{A}_m(\mathbf{r}, t)] &= \hat{D}_s^+(\mathbf{r}_j, t) \left[\left(\frac{1}{2} \bar{\mathbf{J}}^j - i \hat{\mathbf{J}}^j \times \right) \hat{\mathbf{D}}^-(\mathbf{r}_j, t) \right]; \hat{A}_m(\mathbf{r}, t) \\ &= \frac{1}{2} \frac{i\hbar}{2\epsilon(\mathbf{r})} \hat{D}_s^+(\mathbf{r}_j, t) \delta_{sm}^T(\mathbf{r}_j, \mathbf{r}) - i \hat{D}_s^+(\mathbf{r}_j, t) [(\hat{\mathbf{J}}^j \times \hat{\mathbf{D}}^-(\mathbf{r}_j, t))_s; \hat{A}_m(\mathbf{r}, t)] \\ &= \frac{1}{2} \frac{i\hbar}{2\epsilon(\mathbf{r})} \hat{D}_s^+(\mathbf{r}_j, t) \delta_{sm}^T(\mathbf{r}_j, \mathbf{r}) - i \hat{D}_s^+(\mathbf{r}_j, t) [\epsilon_{skl} \hat{J}_k^j \hat{D}_l^-(\mathbf{r}_j, t); \hat{A}_m(\mathbf{r}, t)] \\ &= \frac{1}{2} \frac{i\hbar}{2\epsilon(\mathbf{r})} \delta_{sm}^T(\mathbf{r}_j, \mathbf{r}) \hat{D}_s^+(\mathbf{r}_j, t) + i \epsilon_{lks} \hat{J}_k^j \hat{D}_s^+(\mathbf{r}_j, t) \frac{i\hbar}{2} \delta_{lm}^T(\mathbf{r}_j, \mathbf{r}) \\ &= \frac{i\hbar}{2} \delta_{sm}^T(\mathbf{r}_j, \mathbf{r}) \left(\frac{1}{2} \hat{D}_s^+(\mathbf{r}_j, t) + i (\hat{\mathbf{J}}^j \times \hat{\mathbf{D}}^+(\mathbf{r}_j, t))_s \right) \end{aligned} \quad (\text{C.6.4})$$

Inserting this result into equation (C.6.3) we find the following:

$$\begin{aligned} \frac{\partial}{\partial t} (\nabla \times \hat{\mathbf{A}}(\mathbf{r}, t))_r &= -\frac{1}{\epsilon_0} (\nabla \times \hat{\mathbf{D}}(\mathbf{r}, t))_r - \epsilon_{rnm} \frac{\partial}{\partial r_n} \frac{i}{\hbar} \sum_j^{\text{atoms}} \frac{2\beta}{\epsilon_0} \left\{ \frac{i\hbar}{2} \delta_{ms}^T(\mathbf{r}, \mathbf{r}_j) (\bar{\alpha}_j \hat{\mathbf{D}}^-(\mathbf{r}_j, t))_s \right. \\ &\quad \left. + \frac{i\hbar}{2} \delta_{sm}^T(\mathbf{r}_j, \mathbf{r}) \left(\frac{1}{2} \hat{D}_s^+(\mathbf{r}_j, t) + i (\hat{\mathbf{J}}^j \times \hat{\mathbf{D}}^+(\mathbf{r}_j, t))_s \right) \right\} \\ &= -\frac{1}{\epsilon_0} (\nabla \times \hat{\mathbf{D}}(\mathbf{r}, t))_r - \epsilon_{rnm} \frac{\partial}{\partial r_n} \frac{i}{\hbar} \sum_j^{\text{atoms}} \frac{2\beta}{\epsilon_0} \frac{i\hbar}{2} \delta_{ms}^T(\mathbf{r}, \mathbf{r}_j) \left\{ \frac{1}{2} \hat{D}_s(\mathbf{r}_j, t) \right. \\ &\quad \left. + i (\hat{\mathbf{J}}^j \times (\hat{\mathbf{D}}^+(\mathbf{r}_j, t) - \hat{\mathbf{D}}^-(\mathbf{r}_j, t)))_s \right\} \end{aligned} \quad (\text{C.6.5})$$

On vector form the equation (C.6.5) may be written as:

$$\frac{\partial}{\partial t} \hat{\mathbf{B}}(\mathbf{r}, t) = -\frac{1}{\epsilon_0} \nabla \times \hat{\mathbf{D}}(\mathbf{r}, t) + \sum_j^{\text{atoms}} \frac{\beta}{\epsilon_0} \nabla \times \bar{\delta}^T(\mathbf{r}, \mathbf{r}_j) \cdot \left\{ \frac{1}{2} \hat{\mathbf{D}}(\mathbf{r}_j, t) + i \hat{\mathbf{J}}^j \times (\hat{\mathbf{D}}^+(\mathbf{r}_j, t) - \hat{\mathbf{D}}^-(\mathbf{r}_j, t)) \right\} \quad (\text{C.6.6})$$

C.6.3 Remarks on the atomic spin equation (5.2.1)

Using *Heisenberg equation of motion* we derive the following equation of motion for the spin of the atoms, concerning the r -th vector component of the spin of the k -th atom.

$$\begin{aligned}
\frac{d}{dt} \hat{J}_r^k &= \frac{i}{\hbar} [\hat{\mathcal{H}}_{\text{Eff}}; \hat{J}_r^k] \\
&= -\frac{2\beta}{\hbar\epsilon_0} \sum_j^{\text{atoms}} [\hat{\mathbf{D}}^+(\mathbf{r}_j, t) \cdot (\hat{\mathbf{J}}^j \times \hat{\mathbf{D}}^-(\mathbf{r}_j, t)); \hat{J}_r^k] \\
&= -\frac{2\beta}{\hbar\epsilon_0} \sum_j^{\text{atoms}} [\hat{D}_s^+(\mathbf{r}_j, t) \epsilon_{snm} \hat{J}_n^j \hat{D}_m^-(\mathbf{r}_j, t); \hat{J}_r^k] \\
&= -\frac{2\beta}{\hbar\epsilon_0} \sum_j^{\text{atoms}} \hat{D}_s^+(\mathbf{r}_j, t) \epsilon_{snm} \hat{D}_m^-(\mathbf{r}_j, t) [\hat{J}_n^j; \hat{J}_r^k]
\end{aligned}$$

Using the commutation relation defined in equation (2.3.6) we find:

$$\begin{aligned}
&= -\frac{2\beta}{\hbar\epsilon_0} \sum_j^{\text{atoms}} \hat{D}_s^+(\mathbf{r}_j, t) \epsilon_{snm} \hat{D}_m^-(\mathbf{r}_j, t) i \epsilon_{nrl} \hat{J}_l^j \delta_{jk} \\
&= \frac{2i\beta}{\hbar\epsilon_0} \epsilon_{rln} \hat{J}_l^k \epsilon_{nsm} \hat{D}_s^+(\mathbf{r}_k, t) \hat{D}_m^-(\mathbf{r}_k, t) \\
&= \frac{2i\beta}{\hbar\epsilon_0} (\hat{\mathbf{J}}^k \times \hat{\mathbf{D}}^+(\mathbf{r}_k, t) \times \hat{\mathbf{D}}^-(\mathbf{r}_k, t))_r. \tag{C.6.7}
\end{aligned}$$

On vector form this equation reads:

$$\frac{d}{dt} \hat{\mathbf{J}}^k = \frac{2i\beta}{\hbar\epsilon_0} \hat{\mathbf{J}}^k \times \hat{\mathbf{D}}^+(\mathbf{r}_k, t) \times \hat{\mathbf{D}}^-(\mathbf{r}_k, t). \tag{C.6.8}$$

C.7 Remarks on the general solution using Green's functions

We rewrite the term (5.3.34b) using the explicit expression for the Green's function (5.3.16). This rewriting requires the following considerations:

$$\iint_{t_0}^{t^+} d^3 r' dt' -c^2 (\psi_s^+(\mathbf{r}', t') \nabla'^2 G_{rs}^+(\mathbf{r}, t | \mathbf{r}', t') - G_{rs}^+(\mathbf{r}, t | \mathbf{r}', t') \nabla'^2 \psi_s^+(\mathbf{r}', t'))$$

Including the component form of the explicit expression for the Green's function, we find:

$$\sum_n \frac{1}{\omega_n^2} \iint_{t_0}^{t^+} d^3 r' dt' - c^2 (\psi_s^+(\mathbf{r}', t') \nabla'^2 F_{nr}(\mathbf{r}, t) F_{ns}^*(\mathbf{r}', t') - F_{nr}(\mathbf{r}, t) F_{ns}^*(\mathbf{r}', t') \nabla'^2 \psi_s^+(\mathbf{r}', t'))$$

We then use that the Laplace operator ∇'^2 vanish when working on a field component that is only a function of \mathbf{r} . Using the product-rule of differentiation, the above reduce to:

$$\sum_n \iint_{t_0}^{t^+} d^3 r' dt' - c^2 (\psi_s^+(\mathbf{r}', t') \nabla'^2 F_{ns}^*(\mathbf{r}', t') - F_{ns}^*(\mathbf{r}', t') \nabla'^2 \psi_s^+(\mathbf{r}', t')) F_{nr}(\mathbf{r}, t)$$

This is the component form of equation (5.3.37).

C.8 Remarks on Gauss' Theorem

Vector identities used in Gauss' Theorem 1'st version

In the first version of Gauss' theorem we use the vector identity written below. To shorten notation we suppress all space dependence.

$$\begin{aligned} \nabla \cdot (U \nabla V) &= \frac{\partial}{\partial r_n} \left(U \frac{\partial}{\partial r_n} V \right) \\ &= \frac{\partial}{\partial r_n} U \frac{\partial}{\partial r_n} V + U \frac{\partial^2}{\partial r_n^2} V \\ &= \nabla U \cdot \nabla V + U \nabla^2 V \end{aligned}$$

This identity is used in the first version of Gauss' theorem

Vector identities used in Gauss' Theorem 2'nd version

In the second version of Gauss' theorem we need a few vector identities. \mathbf{n} is a vector not a vector field, whereas both \mathbf{E} and \mathbf{F} are vector fields.

$$\begin{aligned}
 [\mathbf{E} \times (\nabla \times \mathbf{F})] \cdot \mathbf{n} &= \varepsilon_{nml} E_n \varepsilon_{lrs} \frac{\partial}{\partial r_r} F_s \\
 &= \varepsilon_{lmn} n_m E_n \varepsilon_{lrs} \frac{\partial}{\partial r_r} F_s = (\mathbf{n} \times \mathbf{E}) \cdot (\nabla \times \mathbf{F}) \\
 &= E_n \varepsilon_{nlm} \frac{\partial}{\partial r_r} \varepsilon_{lrs} \frac{\partial}{\partial r_r} F_s n_m = \mathbf{E} \cdot [(\nabla \times \mathbf{F}) \times \mathbf{n}] \\
 \nabla \cdot (\mathbf{E} \nabla \cdot \mathbf{F}) &= \frac{\partial}{\partial r_n} \left(E_n \frac{\partial}{\partial r_l} F_l \right) \\
 &= \frac{\partial}{\partial r_n} E_n \frac{\partial}{\partial r_l} F_l + E_n \frac{\partial}{\partial r_n} \frac{\partial}{\partial r_l} F_l \\
 &= (\nabla \cdot \mathbf{E})(\nabla \cdot \mathbf{F}) + \mathbf{E} \cdot \nabla(\nabla \cdot \mathbf{F}) \\
 \nabla \cdot (\mathbf{E} \times [\nabla \times \mathbf{F}]) &= \frac{d}{\partial r_n} \left(\varepsilon_{nml} E_m \varepsilon_{lrs} \frac{\partial}{\partial r_r} F_s \right) \\
 &= \varepsilon_{lmn} \frac{\partial}{\partial r_n} E_m \varepsilon_{lrs} \frac{\partial}{\partial r_r} F_s - E_m \varepsilon_{mnl} \frac{\partial}{\partial r_n} \varepsilon_{lrs} \frac{\partial}{\partial r_r} F_s \\
 &= (\nabla \times \mathbf{E}) \cdot (\nabla \times \mathbf{F}) - \mathbf{E} \cdot \nabla \times (\nabla \times \mathbf{F}).
 \end{aligned}$$

C.8.1 Derivation of Gauss' Theorem 2'nd version

Let us consider the vector field:

$$\mathbf{E} \nabla \cdot \mathbf{F} - \mathbf{F} \nabla \cdot \mathbf{E} + \mathbf{E} \times (\nabla \times \mathbf{F}) - \mathbf{F} \times (\nabla \times \mathbf{E})$$

Taking the divergence of this vector, and using the vector identities derived in the previous section, we find:

$$\mathbf{E} \cdot \nabla^2 \mathbf{F} - \mathbf{F} \cdot \nabla^2 \mathbf{E}$$

On the other hand we find by dotting the vectorfield by the vector \mathbf{n} :

$$[\mathbf{E} \nabla \cdot \mathbf{F} - \mathbf{F} \nabla \cdot \mathbf{E}] \cdot \mathbf{n} - [\mathbf{E} \cdot (\mathbf{n} \times (\nabla \times \mathbf{F})) + (\nabla \times \mathbf{E}) \cdot (\mathbf{n} \times \mathbf{F})]$$

Inserting these identities into Gauss' theorem we find the second version of Gauss' theorem, equation (5.3.36).

Detailed calculations, Part III

D.1 Remarks on equation (6.4.11)

Let us define the following quantity.

$$\gamma = \frac{2\beta^2 k_L^3 \epsilon(\mathbf{r})}{\hbar \epsilon_0 3\pi} \quad (\text{D.1.1})$$

Using shortest possible notation we calculate:

$$\begin{aligned} & i\gamma \hat{\mathbf{J}}_0 \times \{ [\hat{\mathbf{J}}_0 \times \tilde{\mathbf{D}}_0^+] \times \tilde{\mathbf{D}}_0^- + \tilde{\mathbf{D}}_0^+ \times [\hat{\mathbf{J}}_0 \times \tilde{\mathbf{D}}_0^-] \} \\ &= i\gamma \hat{\mathbf{J}}_0 \times \left\{ \begin{pmatrix} \hat{J}_y \tilde{D}_z^+ - \hat{J}_z \tilde{D}_y^+ \\ \hat{J}_z \tilde{D}_x^+ - \hat{J}_x \tilde{D}_z^+ \\ \hat{J}_x \tilde{D}_y^+ - \hat{J}_y \tilde{D}_x^+ \end{pmatrix} \times \begin{pmatrix} \tilde{D}_x^- \\ \tilde{D}_y^- \\ \tilde{D}_z^- \end{pmatrix} + \begin{pmatrix} \tilde{D}_x^+ \\ \tilde{D}_y^+ \\ \tilde{D}_z^+ \end{pmatrix} + \begin{pmatrix} \hat{J}_y \tilde{D}_z^- - \hat{J}_z \tilde{D}_y^- \\ \hat{J}_z \tilde{D}_x^- - \hat{J}_x \tilde{D}_z^- \\ \hat{J}_x \tilde{D}_y^- - \hat{J}_y \tilde{D}_x^- \end{pmatrix} \right\} \\ &= i\gamma \hat{\mathbf{J}}_0 \times \begin{pmatrix} \hat{J}_y (\tilde{D}_x^+ \tilde{D}_y^- - \tilde{D}_y^+ \tilde{D}_x^-) + \hat{J}_z (\tilde{D}_x^+ \tilde{D}_z^- - \tilde{D}_z^+ \tilde{D}_x^-) \\ \hat{J}_x (\tilde{D}_y^+ \tilde{D}_x^- - \tilde{D}_x^+ \tilde{D}_y^-) + \hat{J}_z (\tilde{D}_y^+ \tilde{D}_z^- - \tilde{D}_z^+ \tilde{D}_y^-) \\ \hat{J}_x (\tilde{D}_z^+ \tilde{D}_x^- - \tilde{D}_x^+ \tilde{D}_z^-) + \hat{J}_y (\tilde{D}_z^+ \tilde{D}_y^- - \tilde{D}_y^+ \tilde{D}_z^-) \end{pmatrix} \\ &= i\gamma \begin{pmatrix} \hat{J}_y \hat{J}_x (\tilde{D}_z^+ \tilde{D}_x^- - \tilde{D}_x^+ \tilde{D}_z^-) + \hat{J}_y^2 (\tilde{D}_z^+ \tilde{D}_y^- - \tilde{D}_y^+ \tilde{D}_z^-) - \hat{J}_z \hat{J}_x (\tilde{D}_y^+ \tilde{D}_x^- - \tilde{D}_x^+ \tilde{D}_y^-) - \hat{J}_z^2 (\tilde{D}_y^+ \tilde{D}_z^- - \tilde{D}_z^+ \tilde{D}_y^-) \\ \hat{J}_z \hat{J}_y (\tilde{D}_x^+ \tilde{D}_y^- - \tilde{D}_y^+ \tilde{D}_x^-) + \hat{J}_z^2 (\tilde{D}_x^+ \tilde{D}_z^- - \tilde{D}_z^+ \tilde{D}_x^-) - \hat{J}_x \hat{J}_y (\tilde{D}_z^+ \tilde{D}_y^- - \tilde{D}_y^+ \tilde{D}_z^-) \\ \hat{J}_x^2 (\tilde{D}_y^+ \tilde{D}_x^- - \tilde{D}_x^+ \tilde{D}_y^-) + \hat{J}_x \hat{J}_z (\tilde{D}_y^+ \tilde{D}_z^- - \tilde{D}_z^+ \tilde{D}_y^-) - \hat{J}_y^2 (\tilde{D}_x^+ \tilde{D}_y^- - \tilde{D}_y^+ \tilde{D}_x^-) - \hat{J}_y \hat{J}_z (\tilde{D}_x^+ \tilde{D}_z^- - \tilde{D}_z^+ \tilde{D}_x^-) \end{pmatrix} \end{aligned}$$

Here we use the spin products written in equation (6.4.10).

$$=i\gamma \left(\begin{array}{l} \frac{-i}{2}\hat{J}_z(\tilde{D}_z^+\tilde{D}_x^- - \tilde{D}_x^+\tilde{D}_z^-) - \frac{-i}{2}\hat{J}_y(\tilde{D}_y^+\tilde{D}_x^- - \tilde{D}_x^+\tilde{D}_y^-) + \frac{1}{2}(\tilde{D}_z^+\tilde{D}_y^- - \tilde{D}_y^+\tilde{D}_z^-) \\ \frac{-i}{2}\hat{J}_x(\tilde{D}_x^+\tilde{D}_y^- - \tilde{D}_y^+\tilde{D}_x^-) - \frac{-i}{2}\hat{J}_z(\tilde{D}_z^+\tilde{D}_y^- - \tilde{D}_y^+\tilde{D}_z^-) + \frac{1}{2}(\tilde{D}_x^+\tilde{D}_z^- - \tilde{D}_z^+\tilde{D}_x^-) \\ \frac{-i}{2}\hat{J}_y(\tilde{D}_y^+\tilde{D}_z^- - \tilde{D}_z^+\tilde{D}_y^-) - \frac{-i}{2}\hat{J}_x(\tilde{D}_x^+\tilde{D}_z^- - \tilde{D}_z^+\tilde{D}_x^-) + \frac{1}{2}(\tilde{D}_y^+\tilde{D}_x^- - \tilde{D}_x^+\tilde{D}_y^-) \end{array} \right)$$

This equation is exactly equal equation (6.4.11).

D.2 Remarks on the simple terms in equation (6.4.12)

Below is written the complete expression of equation (6.4.12) in terms of its vector components. The terms are all brought onto ordered form. Since the amount of terms are quite overwhelming we will consider the resulting vector one component at the time. Equation (6.4.12) may thus be written as:

$$-\frac{4\beta^2}{\hbar^2\epsilon_0^2} \int_{t_0}^t dt' \left(\begin{array}{l} x - \text{component} \\ y - \text{component} \\ z - \text{component} \end{array} \right) \quad (\text{D.2.1})$$

The individual components are listed below.

x-component:

$$\begin{aligned} & -\hat{J}_{0x} \left\{ \left[\tilde{D}_x^+(t')\tilde{D}_x^+(t)\tilde{D}_y^-(t')\tilde{D}_y^-(t) + \tilde{D}_y^+(t)\tilde{D}_y^+(t')\tilde{D}_x^-(t)\tilde{D}_x^-(t') \right] \right. \\ & \quad - \left[\tilde{D}_x^+(t')\tilde{D}_y^+(t)\tilde{D}_y^-(t')\tilde{D}_x^-(t) + \tilde{D}_x^+(t)\tilde{D}_y^+(t')\tilde{D}_y^-(t)\tilde{D}_x^-(t') \right] \\ & \quad + \left[\tilde{D}_z^+(t')\tilde{D}_z^+(t)\tilde{D}_x^-(t')\tilde{D}_x^-(t) + \tilde{D}_x^+(t)\tilde{D}_x^+(t')\tilde{D}_z^-(t)\tilde{D}_z^-(t') \right] \\ & \quad \left. - \left[\tilde{D}_z^+(t')\tilde{D}_x^+(t)\tilde{D}_x^-(t')\tilde{D}_z^-(t) + \tilde{D}_z^+(t)\tilde{D}_x^+(t')\tilde{D}_x^-(t)\tilde{D}_z^-(t') \right] \right\} \\ & + \hat{J}_{0y} \left\{ \left[\tilde{D}_y^+(t')\tilde{D}_z^+(t)\tilde{D}_z^-(t')\tilde{D}_x^-(t) + \tilde{D}_x^+(t)\tilde{D}_z^+(t')\tilde{D}_z^-(t)\tilde{D}_y^-(t') \right] \right. \\ & \quad \left. - \left[\tilde{D}_y^+(t')\tilde{D}_x^+(t)\tilde{D}_z^-(t')\tilde{D}_z^-(t) + \tilde{D}_z^+(t)\tilde{D}_z^+(t')\tilde{D}_x^-(t)\tilde{D}_y^-(t') \right] \right\} \\ & + \hat{J}_{z0} \left\{ \left[\tilde{D}_y^+(t')\tilde{D}_x^+(t)\tilde{D}_z^-(t')\tilde{D}_y^-(t) + \tilde{D}_y^+(t)\tilde{D}_z^+(t')\tilde{D}_x^-(t)\tilde{D}_y^-(t') \right] \right. \\ & \quad \left. - \left[\tilde{D}_y^+(t')\tilde{D}_y^+(t)\tilde{D}_z^-(t')\tilde{D}_x^-(t) + \tilde{D}_x^+(t)\tilde{D}_z^+(t')\tilde{D}_y^-(t)\tilde{D}_y^-(t') \right] \right\} \quad (\text{D.2.2}) \end{aligned}$$

y-component

$$\begin{aligned}
& +\hat{J}_{0x} \left\{ \left[\tilde{D}_z^+(t') \tilde{D}_y^+(t) \tilde{D}_x^-(t') \tilde{D}_z^-(t) + \tilde{D}_z^+(t) \tilde{D}_x^+(t') \tilde{D}_y^-(t) \tilde{D}_z^-(t') \right] \right. \\
& \quad \left. - \left[\tilde{D}_z^+(t') \tilde{D}_z^+(t) \tilde{D}_x^-(t') \tilde{D}_y^-(t) + \tilde{D}_y^+(t) \tilde{D}_x^+(t') \tilde{D}_z^-(t) \tilde{D}_z^-(t') \right] \right\} \\
& -\hat{J}_{0y} \left\{ \left[\tilde{D}_y^+(t') \tilde{D}_y^+(t) \tilde{D}_z^-(t') \tilde{D}_z^-(t) + \tilde{D}_z^+(t) \tilde{D}_z^+(t') \tilde{D}_y^-(t) \tilde{D}_y^-(t') \right] \right. \\
& \quad - \left[\tilde{D}_y^+(t') \tilde{D}_z^+(t) \tilde{D}_z^-(t') \tilde{D}_y^-(t) + \tilde{D}_y^+(t) \tilde{D}_z^+(t') \tilde{D}_z^-(t) \tilde{D}_y^-(t') \right] \\
& \quad + \left[\tilde{D}_x^+(t') \tilde{D}_x^+(t) \tilde{D}_y^-(t') \tilde{D}_y^-(t) + \tilde{D}_y^+(t) \tilde{D}_y^+(t') \tilde{D}_x^-(t) \tilde{D}_x^-(t') \right] \\
& \quad \left. - \left[\tilde{D}_x^+(t') \tilde{D}_y^+(t) \tilde{D}_y^-(t') \tilde{D}_x^-(t) + \tilde{D}_x^+(t) \tilde{D}_y^+(t') \tilde{D}_y^-(t) \tilde{D}_x^-(t') \right] \right\} \\
& +\hat{J}_{z0} \left\{ \left[\tilde{D}_z^+(t') \tilde{D}_x^+(t) \tilde{D}_x^-(t') \tilde{D}_y^-(t) + \tilde{D}_y^+(t) \tilde{D}_x^+(t') \tilde{D}_x^-(t) \tilde{D}_z^-(t') \right] \right. \\
& \quad \left. - \left[\tilde{D}_z^+(t') \tilde{D}_y^+(t) \tilde{D}_x^-(t') \tilde{D}_x^-(t) + \tilde{D}_x^+(t) \tilde{D}_x^+(t') \tilde{D}_y^-(t) \tilde{D}_z^-(t') \right] \right\} \tag{D.2.3}
\end{aligned}$$

z-component

$$\begin{aligned}
& +\hat{J}_{0x} \left\{ \left[\tilde{D}_x^+(t') \tilde{D}_y^+(t) \tilde{D}_y^-(t') \tilde{D}_z^-(t) + \tilde{D}_z^+(t) \tilde{D}_y^+(t') \tilde{D}_y^-(t) \tilde{D}_x^-(t') \right] \right. \\
& \quad \left. - \left[\tilde{D}_x^+(t') \tilde{D}_z^+(t) \tilde{D}_y^-(t') \tilde{D}_y^-(t) + \tilde{D}_y^+(t) \tilde{D}_y^+(t') \tilde{D}_z^-(t) \tilde{D}_x^-(t') \right] \right\} \\
& +\hat{J}_{y0} \left\{ \left[\tilde{D}_x^+(t') \tilde{D}_z^+(t) \tilde{D}_y^-(t') \tilde{D}_x^-(t) + \tilde{D}_x^+(t) \tilde{D}_y^+(t') \tilde{D}_z^-(t) \tilde{D}_x^-(t') \right] \right. \\
& \quad \left. - \left[\tilde{D}_x^+(t') \tilde{D}_x^+(t) \tilde{D}_y^-(t') \tilde{D}_z^-(t) + \tilde{D}_z^+(t) \tilde{D}_y^+(t') \tilde{D}_x^-(t) \tilde{D}_x^-(t') \right] \right\} \\
& -\hat{J}_{0z} \left\{ \left[\tilde{D}_z^+(t') \tilde{D}_z^+(t) \tilde{D}_x^-(t') \tilde{D}_x^-(t) + \tilde{D}_x^+(t) \tilde{D}_x^+(t') \tilde{D}_z^-(t) \tilde{D}_z^-(t') \right] \right. \\
& \quad - \left[\tilde{D}_z^+(t') \tilde{D}_x^+(t) \tilde{D}_x^-(t') \tilde{D}_z^-(t) + \tilde{D}_z^+(t) \tilde{D}_x^+(t') \tilde{D}_x^-(t) \tilde{D}_z^-(t') \right] \\
& \quad + \left[\tilde{D}_y^+(t') \tilde{D}_y^+(t) \tilde{D}_z^-(t') \tilde{D}_z^-(t) + \tilde{D}_z^+(t) \tilde{D}_z^+(t') \tilde{D}_y^-(t) \tilde{D}_y^-(t') \right] \\
& \quad \left. - \left[\tilde{D}_y^+(t') \tilde{D}_z^+(t) \tilde{D}_z^-(t') \tilde{D}_y^-(t) + \tilde{D}_y^+(t) \tilde{D}_z^+(t') \tilde{D}_z^-(t) \tilde{D}_y^-(t') \right] \right\} \tag{D.2.4}
\end{aligned}$$

D.3 Remarks on equation (7.2.4)

D.3.1 Local Stokes Operators

Let us just define the general local Stokes Operators.

$$\hat{S}_x = \frac{1}{2}(a_1^\dagger \hat{a}_1 - a_2^\dagger \hat{a}_2) \quad (\text{D.3.1})$$

$$\hat{S}_y = \frac{1}{2}(a_1^\dagger \hat{a}_2 + a_2^\dagger \hat{a}_1) \quad (\text{D.3.2})$$

$$\hat{S}_z = \frac{1}{2i}(a_1^\dagger \hat{a}_2 - a_2^\dagger \hat{a}_1) \quad (\text{D.3.3})$$

We mean local in the respect that the unit vectors \mathbf{e}_s are only locally fixed. We will however use the index 1, 2, 3 or x, y, z at whim. Let us then calculate on equation (7.2.3).

$$\begin{aligned} \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} & \frac{2i\beta}{\hbar\epsilon_0} \bar{\mathbf{J}}_0(\mathbf{r}) \times \tilde{\mathbf{D}}_0^+(\mathbf{r}, t) \times \tilde{\mathbf{D}}_0^-(\mathbf{r}, t) \\ & = \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \frac{2i\beta}{\hbar\epsilon_0} \frac{\hbar\omega_L \epsilon_0}{2} |u_{\mathbf{k}}(\mathbf{r}, t)|^2 \bar{\mathbf{J}} \times \left(\hat{a}_{\mathbf{k}}^\dagger \mathbf{f}_{\mathbf{k}}(\mathbf{r})^* \times \hat{a}_{\mathbf{k}} \mathbf{f}_{\mathbf{k}}(\mathbf{r}) \right) \\ & = \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \frac{2i\beta}{\hbar\epsilon_0} \frac{\hbar\omega_L \epsilon_0}{2} |u_{\mathbf{k}}(\mathbf{r}, t)|^2 \frac{\epsilon(\mathbf{r})}{V} \bar{\mathbf{J}} \times \left(\sum_{ss'} \hat{a}_s^\dagger \mathbf{e}_s \times \hat{a}_{s'} \mathbf{e}_{s'} \right) \\ & = \frac{\langle \rho(\mathbf{r}) \rangle_{\text{Mean}} i\beta\omega_L |u_{\mathbf{k}}(\mathbf{r}, t)|^2 \epsilon(\mathbf{r})}{V} \bar{\mathbf{J}} \times \left((\hat{a}_1^\dagger \mathbf{e}_1 + \hat{a}_2^\dagger \mathbf{e}_2) \times (\hat{a}_1 \mathbf{e}_1 + \hat{a}_2 \mathbf{e}_2) \right) \\ & = \frac{\langle \rho(\mathbf{r}) \rangle_{\text{Mean}} i\beta\omega_L |u_{\mathbf{k}}(\mathbf{r}, t)|^2 \epsilon(\mathbf{r})}{V} \bar{\mathbf{J}} \times \mathbf{e}_z (\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1) \\ & = - \frac{\langle \rho(\mathbf{r}) \rangle_{\text{Mean}} 2\beta\omega_L |u_{\mathbf{k}}(\mathbf{r}, t)|^2 \epsilon(\mathbf{r})}{V} \frac{1}{2i} (\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1) \begin{pmatrix} \bar{J}_y \\ -\bar{J}_x \\ 0 \end{pmatrix} \end{aligned} \quad (\text{D.3.4})$$

D.4 Remarks on equation (7.2.6)

We start by considering equation (7.2.5).

$$\begin{aligned}
& \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \frac{2\beta^2 k_L^3 \epsilon(\mathbf{r})}{\hbar \epsilon_0 3\pi} \left\{ \left(\begin{array}{c} -\bar{J}_y \frac{1}{2} (\tilde{D}_y^+ \tilde{D}_x^- + \tilde{D}_x^+ \tilde{D}_y^-) \\ -\bar{J}_x \frac{1}{2} (\tilde{D}_x^+ \tilde{D}_y^- + \tilde{D}_y^+ \tilde{D}_x^-) \\ 0 \end{array} \right) + \left(\begin{array}{c} 0 \\ 0 \\ \frac{i}{2} (\tilde{D}_y^+ \tilde{D}_x^- - \tilde{D}_x^+ \tilde{D}_y^-) \end{array} \right) \right\} \\
& - \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \frac{2\beta^2 k_L^3 \epsilon(\mathbf{r})}{\hbar \epsilon_0 3\pi} \left(\begin{array}{c} \bar{J}_x (2\tilde{D}_x^+(t') \tilde{D}_x^-(t) + \tilde{D}_y^+(t') \tilde{D}_y^-(t)) \\ \bar{J}_y (\tilde{D}_x^+(t') \tilde{D}_x^-(t) + 2\tilde{D}_y^+(t') \tilde{D}_y^-(t)) \\ \bar{J}_z (\tilde{D}_x^+(t') \tilde{D}_x^-(t) + \tilde{D}_y^+(t') \tilde{D}_y^-(t)) \end{array} \right) \\
& = \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \frac{2\beta^2 k_L^3 \epsilon(\mathbf{r})}{\hbar \epsilon_0 3\pi} \frac{\hbar \omega_L \epsilon(\mathbf{r}) \epsilon_0}{2} \frac{|u_{\mathbf{k}}(\mathbf{r}, t)|^2}{V} \left\{ \left(\begin{array}{c} -\bar{J}_y \frac{1}{2} (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) \\ -\bar{J}_x \frac{1}{2} (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) \\ 0 \end{array} \right) + \left(\begin{array}{c} 0 \\ 0 \\ \frac{1}{2i} (\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1) \end{array} \right) \right\} \\
& - \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \frac{2\beta^2 k_L^3 \epsilon(\mathbf{r})}{\hbar \epsilon_0 3\pi} \frac{\hbar \omega_L \epsilon(\mathbf{r}) \epsilon_0}{2} \frac{|u_{\mathbf{k}}(\mathbf{r}, t)|^2}{V} \left\{ \left(\begin{array}{c} \bar{J}_x (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) \\ \bar{J}_y (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) \\ \bar{J}_z (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) \end{array} \right) + \left(\begin{array}{c} \bar{J}_x \hat{a}_1^\dagger \hat{a}_1 \\ \bar{J}_y \hat{a}_2^\dagger \hat{a}_2 \\ 0 \end{array} \right) \right\} \\
& = \frac{\langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \beta^2 k_L^3 \epsilon^2(\mathbf{r}) \omega_L}{3\pi} \frac{|u_{\mathbf{k}}(\mathbf{r}, t)|^2}{V} \left\{ \hat{S}_y \left(\begin{array}{c} \bar{J}_y \\ \bar{J}_x \\ 0 \end{array} \right) + \hat{S}_z \left(\begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \right\} \\
& - \frac{\langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \beta^2 k_L^3 \epsilon^2(\mathbf{r}) \omega_L}{3\pi} \frac{|u_{\mathbf{k}}(\mathbf{r}, t)|^2}{V} \left\{ \hat{I}_L \left(\begin{array}{c} \bar{J}_x \\ \bar{J}_y \\ \bar{J}_z \end{array} \right) + \left(\begin{array}{c} \bar{J}_x \hat{a}_1^\dagger \hat{a}_1 \\ \bar{J}_y \hat{a}_2^\dagger \hat{a}_2 \\ 0 \end{array} \right) \right\} \tag{D.4.1}
\end{aligned}$$

D.5 Remarks on equation (7.2.8)

The calculation is a matter of plugging the field components in, and then slight reduction.

$$\begin{aligned}
& - \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \frac{4\beta^2}{\hbar^2 \epsilon_0^2} \int_{t_0}^t dt' \left\{ \left(\begin{array}{c} -\bar{J}_x [\tilde{D}_x^+(t') \tilde{D}_x^+(t) \tilde{D}_y^-(t') \tilde{D}_y^-(t) + \tilde{D}_y^+(t) \tilde{D}_y^+(t') \tilde{D}_x^-(t) \tilde{D}_x^-(t')] \\ -\bar{J}_y [\tilde{D}_x^+(t') \tilde{D}_x^+(t) \tilde{D}_y^-(t') \tilde{D}_y^-(t) + \tilde{D}_y^+(t) \tilde{D}_y^+(t') \tilde{D}_x^-(t) \tilde{D}_x^-(t')] \\ 0 \end{array} \right) \right. \\
& \quad \left. + \left(\begin{array}{c} \bar{J}_x [\tilde{D}_x^+(t') \tilde{D}_y^+(t) \tilde{D}_y^-(t') \tilde{D}_x^-(t) + \tilde{D}_x^+(t) \tilde{D}_y^+(t') \tilde{D}_y^-(t) \tilde{D}_x^-(t')] \\ \bar{J}_y [\tilde{D}_x^+(t') \tilde{D}_y^+(t) \tilde{D}_y^-(t') \tilde{D}_x^-(t) + \tilde{D}_x^+(t) \tilde{D}_y^+(t') \tilde{D}_y^-(t) \tilde{D}_x^-(t')] \\ 0 \end{array} \right) \right\} \\
& = - \langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \frac{4\beta^2}{\hbar^2 \epsilon_0^2} \int_{t_0}^t dt' \left(\frac{\hbar \omega_L \epsilon_0 \epsilon(\mathbf{r})}{4} \right)^2 \frac{|u_{\mathbf{k}}(\mathbf{r}, t)|^2}{V} \frac{|u_{\mathbf{k}}(\mathbf{r}, t')|^2}{V} \left\{ \left(\begin{array}{c} -\bar{J}_x [\hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_2 \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_1 \hat{a}_1] \\ -\bar{J}_y [\hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_2 \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_1 \hat{a}_1] \\ 0 \end{array} \right) \right. \\
& \quad \left. + \left(\begin{array}{c} \bar{J}_x [\hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1] \\ \bar{J}_y [\hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1] \\ 0 \end{array} \right) \right\} \\
& \quad \frac{\langle \rho(\mathbf{r}) \rangle_{\text{Mean}} \beta^2 \epsilon^2(\mathbf{r}) \omega_L^2}{V} \frac{|u_{\mathbf{k}}(\mathbf{r}, t)|^2}{V} \left[\int_0^t dt' |u_{\mathbf{k}}(\mathbf{r}, t')|^2 \right] \left\{ \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1 \right. \\
& \quad \left. - \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_2 \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_1 \hat{a}_1 \right\} \begin{pmatrix} \bar{J}_x \\ \bar{J}_y \\ 0 \end{pmatrix} \quad (\text{D.5.1})
\end{aligned}$$

Let us then consider the operator $\hat{\Gamma}_z^{(2)}$

$$\hat{\Gamma}_z^{(2)} = \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_2 \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_1 \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1. \quad (\text{D.5.2})$$

We see from the calculation:

$$\begin{aligned}
-4\hat{S}_z^2 &= (\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1)(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1) \\
&= \hat{a}_1^\dagger \hat{a}_2 \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2 \hat{a}_2^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_1 \hat{a}_1^\dagger \hat{a}_2
\end{aligned}$$

Using commutation relations applying for the creation and annihilation operators, we may find:

$$\begin{aligned}
&= \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_2 \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_1 \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2 \\
&= \hat{\Gamma}_z^{(2)} - \hat{I}_L, \quad (\text{D.5.3})
\end{aligned}$$

that we may very well set $\hat{\Gamma}_z^{(2)} = -4\hat{S}_z^2 + \hat{I}_L$.

D.6 Calculating the propagator $\bar{P}^+(\mathbf{r}, t|\mathbf{r}', t')$

In the following we make the calculation:

$$\nabla' \times \nabla' \times \bar{G}^+(\mathbf{r}, t|\mathbf{r}', t') = \nabla' \times \nabla' \times \bar{I} \delta\left(t - \left\{t_0 + \frac{\sqrt{\epsilon(\mathbf{r})}|\mathbf{r} - \mathbf{r}_0|}{c}\right\}\right) \frac{e^{ik_L|\mathbf{r} - \mathbf{r}_0|}}{|\mathbf{r} - \mathbf{r}_0|}.$$

We then use the Fourier transformed of the delta function, and find the following useful expression for the Green's function:

$$\bar{G}^+(\mathbf{r}, t|\mathbf{r}', t') = \int d\omega \frac{1}{2\pi} \frac{e^{-i(k-k_L)|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} e^{i\omega(t-t')} \bar{I} \quad (\text{D.6.1})$$

\bar{I} being the identity functional, we may very well set it to:

$$\bar{I} = \mathbf{e}_1 \mathbf{e}_1 + \mathbf{e}_2 \mathbf{e}_2 + \mathbf{e}_3 \mathbf{e}_3, \quad (\text{D.6.2})$$

where the vectors \mathbf{e}_n is an orthonormal basis spanning the three dimensional vector space. From this we see that the interesting differentiation to make are the following:

$$\nabla' \times \nabla' \times \frac{e^{-i(k-k_L)|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \mathbf{n} \quad (\text{D.6.3})$$

\mathbf{n} is some unit vector. We then rewrite the differential operator $\nabla' \times \nabla' \times$, and the calculation then looks like:

$$\nabla' \left(\nabla' \cdot \frac{e^{-i(k-k_L)|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \mathbf{n} \right) - \nabla'^2 \frac{e^{-i(k-k_L)|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \mathbf{n} \quad (\text{D.6.4})$$

The calculation then follows, using the definition of the differential operator ∇' , we defined in Notation and Conventions. After careful differentiation we find.

$$\nabla'^2 \frac{e^{-i(k-k_L)|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \mathbf{n} = - \frac{(k - k_L)^2}{|\mathbf{r} - \mathbf{r}'|} e^{-i(k-k_L)|\mathbf{r} - \mathbf{r}'|} \mathbf{n} \quad (\text{D.6.5a})$$

$$\nabla' \left(\nabla' \cdot \frac{e^{-i(k-k_L)|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \mathbf{n} \right) = \left\{ \left(- \frac{(k - k_L)^2}{|\mathbf{r} - \mathbf{r}'|^3} + - \frac{3i(k - k_L)}{|\mathbf{r} - \mathbf{r}'|^4} - - \frac{3}{|\mathbf{r} - \mathbf{r}'|^5} \right) ((\mathbf{r} - \mathbf{r}') \cdot \mathbf{n})(\mathbf{r} - \mathbf{r}') \right. \\ \left. \left(- \frac{i(k - k_L)}{|\mathbf{r} - \mathbf{r}'|^2} - - \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \right) \mathbf{n} \right\} e^{-i(k-k_L)|\mathbf{r} - \mathbf{r}'|} \quad (\text{D.6.5b})$$

We may combine these two equations to the following result.

$$\begin{aligned} \nabla' \times \nabla' \times \frac{e^{-i(k-k_L)|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \mathbf{n} &= \frac{(k-k_L)^2}{|\mathbf{r}-\mathbf{r}'|^3} (\mathbf{r}-\mathbf{r}') \times (\mathbf{n} \times (\mathbf{r}-\mathbf{r}')) e^{-i(k-k_L)|\mathbf{r}-\mathbf{r}'|} \\ &\quad \left(\frac{i(k-k_L)}{|\mathbf{r}-\mathbf{r}'|^4} + \frac{1}{|\mathbf{r}-\mathbf{r}'|^5} \right) \left[3((\mathbf{r}-\mathbf{r}') \cdot \mathbf{n})(\mathbf{r}-\mathbf{r}') - (\mathbf{r}-\mathbf{r}')^2 \mathbf{n} \right] e^{-i(k-k_L)|\mathbf{r}-\mathbf{r}'|} \end{aligned} \quad (\text{D.6.6})$$

The first term describes the far field, whereas the last term describe the near field. As we have done before we will approximate k with the laser carrier wave number k_L , whenever this is not appearing in an exponential. In this particular case the equation reduce considerably, and we are left with only the near field solution.

$$\nabla' \times \nabla' \times \frac{e^{-i(k-k_L)|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \mathbf{n} = \frac{1}{|\mathbf{r}-\mathbf{r}'|^5} \left[3((\mathbf{r}-\mathbf{r}') \cdot \mathbf{n})(\mathbf{r}-\mathbf{r}') - (\mathbf{r}-\mathbf{r}')^2 \mathbf{n} \right] e^{-i(k-k_L)|\mathbf{r}-\mathbf{r}'|} \quad (\text{D.6.7})$$

Including this result into the Green's function, and doing the Fourier transform again, the propagator $\bar{P}^+(\mathbf{r}, t|\mathbf{r}', t')$ may be written as the following:

$$\begin{aligned} \bar{P}^+(\mathbf{r}, t|\mathbf{r}', t') &= \\ &\sum_{s=1}^3 \frac{1}{|\mathbf{r}-\mathbf{r}'|^5} \left[3((\mathbf{r}-\mathbf{r}') \cdot \mathbf{e}_s)(\mathbf{r}-\mathbf{r}') - (\mathbf{r}-\mathbf{r}')^2 \mathbf{e}_s \right] \mathbf{e}_s e^{ik_L|\mathbf{r}-\mathbf{r}'|} \delta\left(t - \left\{t' + \frac{\sqrt{\epsilon(\mathbf{r})}|\mathbf{r}-\mathbf{r}'|}{c}\right\}\right) \end{aligned} \quad (\text{D.6.8})$$

The calculations regarding the other propagator $\bar{P}^-(\mathbf{r}, t|\mathbf{r}', t')$ are completely equivalent. In this case however we will have slightly more terms since we in this case are dealing with a factor $k+k_L$ instead. Thus we end up with the following propagator.

$$\begin{aligned} \bar{P}^-(\mathbf{r}, t|\mathbf{r}', t') &= \sum_{s=1}^3 \left(\frac{2ik_L}{|\mathbf{r}-\mathbf{r}'|^4} + \frac{1}{|\mathbf{r}-\mathbf{r}'|^5} \right) \left[3((\mathbf{r}-\mathbf{r}') \cdot \mathbf{e}_s)(\mathbf{r}-\mathbf{r}') \right. \\ &\quad \left. - (\mathbf{r}-\mathbf{r}')^2 \mathbf{e}_s \right] \mathbf{e}_s e^{-ik_L|\mathbf{r}-\mathbf{r}'|} \delta\left(t - \left\{t' + \frac{\sqrt{\epsilon(\mathbf{r})}|\mathbf{r}-\mathbf{r}'|}{c}\right\}\right) \\ &\quad + \sum_{s=1}^3 \frac{4k_L^2}{|\mathbf{r}-\mathbf{r}'|^3} (\mathbf{r}-\mathbf{r}') \times (\mathbf{e}_s \times (\mathbf{r}-\mathbf{r}')) \mathbf{e}_s e^{-ik_L|\mathbf{r}-\mathbf{r}'|} \delta\left(t - \left\{t' + \frac{\sqrt{\epsilon(\mathbf{r})}|\mathbf{r}-\mathbf{r}'|}{c}\right\}\right) \end{aligned} \quad (\text{D.6.9})$$

In this case we are both dealing with a near field solution, and a far field solution.

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